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Cut Sets for Picture Fuzzy Sets

Research Article

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ABSTRACT

In today's world, uncertainty is a crucial concept across nearly all areas of business, economics, science, and engineering. Numerous theories have been proposed to manage these uncertain or ambiguous data. Among these, the picture fuzzy set theory stands out as one of the most effective approaches for meticulously addressing vague data. Cut sets hold significant importance in picture fuzzy set theory as they serve as the link between picture fuzzy sets and classical sets, playing a vital role in picture fuzzy arithmetic and decision-making. This article establishes various types of cut sets for picture fuzzy sets and thoroughly explores several relevant properties.

Keywords: *Picture Fuzzy Sets, Level Sets, Cut Sets, Quasi Cut Sets*

1. Introduction

In the real world, many applications contain vaguely defined data values, including sensor information, artificial intelligence, and artificial neural networks. Fuzzy set (FS), which generalizes classical set theory, was introduced by Zadeh [14] in his seminal 1965 paper to manage inexact and imprecise data, acknowledging the presence of vague information across various applications. In 1986, K.T. Atanassov [1] proposed the intuitionistic fuzzy set (IFS) to further address this vagueness by expanding the concept of fuzzy sets. In an intuitionistic fuzzy set, each element is associated with a membership degree and a non-membership degree, with the difference between the sum of these degrees and unity representing the element's hesitant degree. Many applications have found that this hesitant degree can lead to complex situations in decision-making. In 2013, Cuong and Kreinovich [5, 6] introduced the picture fuzzy set (PFS), a robust tool for managing vagueness and uncertainty by incorporating positive, negative, and neutral membership degrees for elements. In this framework, the hesitant degree is split into neutral and refusal membership degrees. Additionally, the cut set in fuzzy set theory is a crucial concept that

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plays an essential role in fuzzy arithmetic. Cut sets play a significant role in fuzzy set theory as they serve as a link between picture fuzzy sets and traditional crisp sets, facilitating the analysis and manipulation of picture fuzzy concepts through classical set theory. They are essential for executing operations on picture fuzzy sets, allowing for the breakdown of picture fuzzy sets into a series of crisp sets, and are a vital resource in various applications. The concept of α-cut set in fuzzy set was initially introduced by L.A. Zadeh [14]. Numerous researchers have identified different types of cut sets [1,2,4,11] for intuitionistic fuzzy sets and have explored many of their properties. Barbhuiya S.R [3] created the quasi cut set for both fuzzy and intuitionistic fuzzy sets and examined various related properties. Hasan et al. [9, 10] defined lower and upper (α,δ,β) −cut sets for picture fuzzy sets. Dutta and Ganju [7] presented the extension principle along with (α,δ,β) −cut and strong (α,δ,β) −cut, as well as level sets for picture fuzzy sets, discussing several of their properties. A variety of properties and applications of picture fuzzy sets are reviewed in [8, 12, 13]. This paper introduces several types of cut sets and quasi cut sets for picture fuzzy sets, discussing many properties associated with these definitions.

2. Preliminaries

In this section, few elementary definitions are discussed that will be used in later sections.

Definition 2.1 ([14]). A FS A in U is given by $A=\{(\theta,\xi_A(\theta)):\theta\in U\}$, with $\xi_A:U\to[0,1]$ and $0\le\xi_A(\theta)\le 1;\forall\alpha\in U$.

Definition 2.2: [14] Let, $A=\{(\theta,\xi_A(\theta)):\theta\in U\}$, be a FS on U and $\alpha\in[0,1]$, then $A^{\alpha}=\{\theta\in U:\xi_A(\theta)\ge\alpha\}$ and $A^{\alpha+}=\{\theta\in U:\xi_A(\theta)>\alpha\}$ are the α −cut of A and the strong α −cut of A respectively.

Definition 2.3 ([1]). An IFS A in U is defined by $A=\{(\theta,\xi_A(\theta),\zeta_A(\theta)):\theta\in U\}$, with $\xi_A:U\to[0,1]$, $\zeta_A:U\to[0,1]$ and $0\le\xi_A(\theta)+\zeta(\theta)\le 1;\forall\theta\in U$ and the hesitant value, $\pi_A(\theta)=1-\left(\xi_A(\theta)+\zeta_A(\theta)\right)$.

Definition 2.4: [1, 2, 11] Let, $A=\{(\theta,\xi_A(\theta),\zeta_A(\theta)):\theta\in U\}\in IFS(U)$ and $\alpha,\beta\in[0,1]$, $\alpha+\beta\le 1$. Then

$$A^{(\alpha,\beta)}=\{\theta:\xi_A(\theta)\ge\alpha\ ,\zeta_A(\theta)\le\beta\}\quad\text{and}$$
$$A^{(\alpha,\beta)+}=\{\theta:\xi_A(\theta)>\alpha\ ,\zeta_A(\theta)<\beta\}$$

are the (α,β) −cut of A and the strong (α,β) −cut of A respectively.

Definition 2.5: [3] Let, $A=\{(\theta,\xi_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be an IFS on U and $\alpha,\beta\in[0,1]$, then the upper quasi −cut of A denoted by $A^{[\alpha,\beta]}$ and is defined by

$$A^{[\alpha,\beta]}=\{\theta\in U:\alpha+\xi_A(\theta)\ge 1\ ,\beta+\zeta_A(\theta)\le 1\}.$$

Definition 2.6: [3] Let, $A=\{(\theta,\xi_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be an IFS on U and $\alpha,\beta\in[0,1]$, then the strong upper quasi −cut set of A denoted by $A^{[\alpha,\beta]+}$ and is defined by

$$A^{[\alpha,\beta]+}=\{\theta\in U:\alpha+\xi_A(\theta)>1\ ,\beta+\zeta_A(\theta)<1\}.$$

Definition 2.7: [3] Let, $A=\{(\theta,\xi_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be an IFS on U and $\alpha,\beta\in[0,1]$, then the lower quasi −cut of A denoted by $A_{[\alpha,\beta]}$ and is defined by

$$A_{[\alpha,\beta]}=\{\theta\in U:\alpha+\xi_A(\theta)\le 1\ ,\beta+\zeta_A(\theta)\ge 1\}.$$

Definition 2.8: [3] Let, $A=\{(\theta,\xi_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be an IFS on U and $\alpha,\beta\in[0,1]$, then the strong lower quasi −cut set of A denoted by $A_{[\alpha,\beta]+}$ and is defined by $A_{[\alpha,\beta]+}=\{\theta\in U:\alpha+\xi_A(\theta)<1\ ,\beta+\zeta_A(\theta)>1\}$.

Definition 2.9: [4, 5] A PFS A on U is given by $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ with $\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)\in[0,1]$ and $0\le\xi_A(\theta)+\tau_A(\theta)+\zeta_A(\theta)\le 1;\forall\theta\in U$ and the refusal value, $\pi_A(\theta)=1-\left(\xi_A(\theta)+\tau_A(\theta)+\zeta_A(\theta)\right);\forall\theta\in U$. In the whole paper, $PFS(U)$ represents the collection of all PFSs in U.

Definition 2.10:[9] Let, $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be a PFS on U and $\alpha,\gamma,\beta\in[0,1]$ with $\alpha+\gamma+\beta\le 1$, then the lower (α,γ,β) −cut of A is given by

$$A_{(\alpha,\gamma,\beta)} = \{\theta \in U: \xi_A(\theta) \leq \alpha , \tau_A(\theta) \leq \gamma , \zeta_A(\theta) \geq \beta\}.$$

That is, $\alpha_{\underline{\xi}_A} = \{\theta \in U: \xi_A(\theta) \leq \alpha\}$, $\gamma_{\underline{\tau}_A} = \{\theta \in U: \tau_A(\theta) \leq \gamma\}$ and $\beta_{\underline{\zeta}_A} = \{\theta \in U: \zeta_A(\theta) \geq \beta\}$ are the lower α, γ and β- cuts of positive membership, neutral membership and negative membership of a PFS A respectively.

Definition 2.11:[9] Let, $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)): \theta \in U\}$ be a PFS on U and $\alpha, \gamma, \beta \in [0,1]$ with $\alpha + \gamma + \beta \leq 1$, then the strong lower (α, γ, β) −cut of A is given by $A_{(\alpha,\gamma,\beta)+} = \{\theta \in U: \xi_A(\theta) < \alpha , \tau_A(\theta) < \gamma , \zeta_A(\theta) > \beta\}$.

That is $\alpha^+{}_{\underline{\xi}_A} = \{\theta \in U: \xi_A(\theta) < \alpha\}$, $\gamma^+{}_{\underline{\tau}_A} = \{\theta \in U: \tau_A(\theta) < \gamma\}$ and $\beta^+{}_{\underline{\zeta}_A} = \{\theta \in U: \zeta_A(\theta) > \beta\}$ are the strong lower α, γ and β- cuts of positive membership, neutral membership and negative membership of a PFS A respectively.

3. Upper (α, γ, β) − Cut Sets for Picture Fuzzy Sets

In this section, we present different types of cut sets for picture fuzzy set. Some related properties are explored. We have followed [2], [4], [6]

Definition 3.1:[10]
Let, $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)): \theta \in U\}$ be a PFS on U and $\alpha, \gamma, \beta \in [0,1]$, $\alpha + \gamma + \beta \leq 1$, then the upper (α, γ, β) −cut of A is given by

$$A^{(\alpha,\gamma,\beta)} = \{\theta \in U: \xi_A(\theta) \geq \alpha , \tau_A(\theta) \geq \gamma , \zeta_A(\theta) \leq \beta\}.$$

That is, $\alpha_{\xi_A} = \{\theta \in U: \xi_A(\theta) \geq \alpha\}$, $\gamma_{\tau_A} = \{\theta \in U: \tau_A(\theta) \geq \gamma\}$ and $\beta_{\zeta_A} = \{\theta \in U: \zeta_A(\theta) \leq \beta\}$ are the upper α, γ and β- cuts of positive membership, neutral membership and negative membership of a PFS A respectively.

Example 3.1: Let, $U = \{a, b, c, d, e\}$ and

$$A = \left\{ \begin{matrix} (a, 0.4,0.2,0.4), (b, 0.3,0.5,0.2), (c, 0.2,0.3,0.4), \\ (d, 0.5, 0.3, 0.2), (e, 0.2, 0.4, 0.4) \end{matrix} \right\}$$

be a PFS in U. Let, $\alpha = 0.3$, $\gamma = 0.2$, $\beta = 0.5$, then

$$A^{(\alpha,\gamma,\beta)} = \{\theta \in U: \xi_A(\theta) \geq 0.3 , \tau_A(\theta) \geq 0.2 , \zeta_A(\theta) \leq 0.5\} = \{a, b, d\}.$$

Definition 3.2: [10] Let, $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)): \theta \in U\}$ be a PFS on U and $\alpha, \gamma, \beta \in [0,1]$, $\alpha + \gamma + \beta \leq 1$, then the strong upper (α, γ, β) −cut of A is given by
$A^{(\alpha,\gamma,\beta)+} = \{\theta \in U: \xi_A(\theta) > \alpha , \tau_A(\theta) > \gamma , \zeta_A(\theta) < \beta\}$.

That is, $\alpha^+_{\xi_A} = \{\theta \in U: \xi_A(\theta) > \alpha\}$, $\gamma^+_{\tau_A} = \{\theta \in U: \tau_A(\theta) > \gamma\}$ and $\beta^+_{\zeta_A} = \{\theta \in U: \zeta_A(\theta) < \beta\}$ are the strong upper α, γ and β- cuts of positive membership, neutral membership and negative membership of a PFS A respectively.

Example 3.2: Let $U = \{a, b, c, d, e\}$ and
$A = \{(a, 0.4,0.2,0.4), (b, 0.3,0.5,0.2), (c, 0.2,0.3,0.4), (d, 0.5, 0.3, 0.2), (e, 0.2, 0.4, 0.4)\}$ be a PFS in U. Let, $\alpha = 0.3$, $\gamma = 0.2$, $\beta = 0.5$, then

$$A^{(\alpha,\gamma,\beta)+} = \{\theta \in U: \xi_A(\theta) > 0.3 , \tau_A(\theta) > 0.2 , \zeta_A(\theta) < 0.5\} = \{d\}.$$

Theorem 3.3: Let, $A \in PFS(U)$, then $A^{(\alpha,\gamma,\beta)+} \subseteq A^{(\alpha,\gamma,\beta)}$.

Proof: Let, $\theta \in A^{(\alpha,\gamma,\beta)+}$, then

$$\xi_A(\theta) > \alpha, \tau_A(\theta) > \gamma , \zeta_A(\theta) < \beta$$
$$\Rightarrow \xi_A(\theta) \geq \alpha , \tau_A(\theta) \geq \gamma , \zeta_A(\theta) \leq \beta$$
$$\Rightarrow \theta \in A^{(\alpha,\gamma,\beta)}.$$
$$\therefore A^{(\alpha,\gamma,\beta)+} \subseteq A^{(\alpha,\gamma,\beta)}.$$

Theorem 3.4: Let, $A, B \in PFS(U)$. If $A \subseteq B$, then

(1). $A^{(\alpha,\gamma,\beta)} \subseteq B^{(\alpha,\gamma,\beta)}$
(2). $A^{(\alpha,\gamma,\beta)+} \subseteq B^{(\alpha,\gamma,\beta)+}$

Proof: (1). Let, $\theta \in A^{(\alpha,\gamma,\beta)}$. Then
$\xi_A(\theta) \geq \alpha , \tau_A(\theta) \geq \gamma , \zeta_A(\theta) \leq \beta$

As $A \subseteq B$, we have

$$\xi_A(\theta) \leq \xi_B(\theta), \tau_A(\theta) \leq \tau_B(\theta) , \zeta_A(\theta) \geq \zeta_B(\theta)$$
$$\Rightarrow \xi_B(\theta) \geq \xi_A(\theta) \geq \alpha , \tau_B(\theta) \geq \tau_A(\theta) \geq \gamma , \zeta_B(\theta) \leq \zeta_A(\theta) \leq \beta$$

$\Rightarrow \xi_B(\theta) \geq \alpha\,, \tau_B(\theta) \geq \gamma\,, \zeta_B(\theta) \leq \beta$

$\Rightarrow\ \theta \in B^{(\alpha,\gamma,\beta)}$.

$\therefore\ A^{(\alpha,\gamma,\beta)} \subseteq B^{(\alpha,\gamma,\beta)}$

(2). Let, $\theta \in A^{(\alpha,\gamma,\beta)+}$. Then $\xi_A(\theta) > \alpha\,, \tau_A(\theta) > \gamma\,, \zeta_A(\theta) < \beta$.

As $A \subseteq B$, we have

$$\xi_A(\theta) \leq \xi_B(\theta), \tau_A(\theta) \leq \tau_B(\theta)\,, \zeta_A(\theta) \geq \zeta_B(\theta)$$

$$\Rightarrow \xi_B(\theta) \geq \xi_A(\theta) > \alpha\,, \tau_B(\theta) \geq \tau_A(\theta) > \gamma\,, \zeta_B(\theta) \leq \zeta_A(\theta) < \beta$$

$\Rightarrow \xi_B(\theta) > \alpha\,, \tau_B(\theta) > \gamma\,, \zeta_B(\theta) < \beta$

$\Rightarrow \theta \in B^{(\alpha,\gamma,\beta)+}$.

$\therefore\ A^{(\alpha,\gamma,\beta)+} \subseteq B^{(\alpha,\gamma,\beta)+}$

Theorem 3.5: Let, $A, B \in PFS(U)$, then $A = B$ implies

(1). $A^{(\alpha,\gamma,\beta)} = B^{(\alpha,\gamma,\beta)}$,
(2). $A^{(\alpha,\gamma,\beta)+} = B^{(\alpha,\gamma,\beta)+}$.

Proof: (1). Let, $\theta \in A^{(\alpha,\gamma,\beta)}$, then $\xi_A(\theta) \geq \alpha\,, \tau_A(\theta) \geq \gamma\,, \zeta_A(\theta) \leq \beta$.

As $A = B$, we have

$$\xi_B(\theta) = \xi_A(\theta) \geq \alpha\,, \tau_B(\theta) = \tau_A(\theta) \geq \gamma\,, \zeta_B(\theta) = \zeta_A(\theta) \leq \beta$$

$\Rightarrow\ \xi_B(\theta) \geq \alpha\,, \tau_B(\theta) \geq \gamma\,, \zeta_B(\theta) \leq \beta$

$\Rightarrow\ \theta \in B^{(\alpha,\gamma,\beta)}$.

$\therefore\ A^{(\alpha,\gamma,\beta)} \subseteq B^{(\alpha,\gamma,\beta)}$.

Again, let $\theta \in B^{(\alpha,\gamma,\beta)}$, then $\xi_B(\theta) \geq \alpha\,, \tau_B(\theta) \geq \gamma\,, \zeta_B(\theta) \leq \beta$.

As $A = B$, we have

$$\xi_A(\theta) = \xi_B(\theta) \geq \alpha\,, \tau_A(\theta) = \tau_B(\theta) \geq \gamma\,, \zeta_A(\theta) = \zeta_B(\theta) \leq \beta$$

$\Rightarrow\ \xi_A(\theta) \geq \alpha\,, \tau_A(\theta) \geq \gamma\,, \zeta_A(\theta) \leq \beta$

$\Rightarrow\ \theta \in A^{(\alpha,\gamma,\beta)}$.

$\therefore\ B^{(\alpha,\gamma,\beta)} \subseteq A^{(\alpha,\gamma,\beta)}$.

Thus, $A^{(\alpha,\gamma,\beta)} = B^{(\alpha,\gamma,\beta)}$.

(2). Let, $\theta \in A^{(\alpha,\gamma,\beta)+}$, then $\xi_A(\theta) > \alpha\,, \tau_A(\theta) > \gamma\,, \zeta_A(\theta) < \beta$.

As $A = B$, we have

$$\xi_B(\theta) = \xi_A(\theta) > \alpha\,, \tau_B(\theta) = \tau_A(\theta) > \gamma\,, \zeta_B(\theta) = \zeta_A(\theta) < \beta$$

$\Rightarrow\ \xi_B(\theta) > \alpha\,, \tau_B(\theta) > \gamma\,, \zeta_B(\theta) < \beta$

$\Rightarrow\ \theta \in B^{(\alpha,\gamma,\beta)+}$.

$\therefore\ A^{(\alpha,\gamma,\beta)+} \subseteq B^{(\alpha,\gamma,\beta)+}$.

Again, let $\theta \in B^{(\alpha,\gamma,\beta)+}$, then $\xi_B(\theta) > \alpha\,, \tau_B(\theta) > \gamma\,, \zeta_B(\theta) < \beta$.

As $A = B$, we have

$$\xi_A(\theta) = \xi_B(\theta) > \alpha\,, \tau_A(\theta) = \tau_B(\theta) > \gamma\,, \zeta_A(\theta) = \zeta_B(\theta) < \beta$$

$\Rightarrow\ \xi_A(\theta) > \alpha\,, \tau_A(\theta) > \gamma\,, \zeta_A(\theta) < \beta$

$\Rightarrow\ \theta \in A^{(\alpha,\gamma,\beta)+}$.

$\therefore\ B^{(\alpha,\gamma,\beta)+} \subseteq A^{(\alpha,\gamma,\beta)+}$.

Thus, $A^{(\alpha,\gamma,\beta)+} = B^{(\alpha,\gamma,\beta)+}$.

Theorem 3.6: Let, $A, B \in PFS(U)$, then

(1). $(A \cap B)^{(\alpha,\gamma,\beta)} = A^{(\alpha,\gamma,\beta)} \cap B^{(\alpha,\gamma,\beta)}$
(2). $(A \cap B)^{(\alpha,\gamma,\beta)+} = A^{(\alpha,\gamma,\beta)+} \cap B^{(\alpha,\gamma,\beta)+}$

Proof: (1). We have, $A \cap B \subseteq A$ and $A \cap B \subseteq B$

Hence from Theorem 3.4, we have

$(A \cap B)^{(\alpha,\gamma,\beta)} \subseteq A^{(\alpha,\gamma,\beta)}$ and $(A \cap B)^{(\alpha,\gamma,\beta)} \subseteq B^{(\alpha,\gamma,\beta)}$

$\Rightarrow (A \cap B)^{(\alpha,\gamma,\beta)} \subseteq A^{(\alpha,\gamma,\beta)} \cap B^{(\alpha,\gamma,\beta)}$

Again, let $\theta \in A^{(\alpha,\gamma,\beta)} \cap B^{(\alpha,\gamma,\beta)}$

$\Rightarrow \theta \in A^{(\alpha,\gamma,\beta)}$ and $\theta \in B^{(\alpha,\gamma,\beta)}$

$\xi_A(\theta) \geq \alpha\,, \xi_B(\theta) \geq \alpha\ \Rightarrow min\{\xi_A(\theta), \xi_B(\theta)\} \geq \alpha$

$\tau_A(\theta) \geq \gamma, \tau_B(\theta) \geq \gamma\ \Rightarrow min\{\tau_A(\theta), \tau_B(\theta)\} \geq \gamma$

$\zeta_A(\theta) \leq \beta, \zeta_B(\theta) \leq \beta\ \Rightarrow max\{\zeta_A(\theta), \zeta_B(\theta)\} \leq \beta$

$\Rightarrow \theta \in (A \cap B)^{(\alpha,\gamma,\beta)}$

$\Rightarrow A^{(\alpha,\gamma,\beta)} \cap B^{(\alpha,\gamma,\beta)} \subseteq (A \cap B)^{(\alpha,\gamma,\beta)}$

Hence, $(A \cap B)^{(\alpha,\gamma,\beta)} = A^{(\alpha,\gamma,\beta)} \cap B^{(\alpha,\gamma,\beta)}$

(2). We have, $A \cap B \subseteq A$ and $A \cap B \subseteq B$

Hence from Theorem 3.4, we have

$(A \cap B)^{(\alpha,\gamma,\beta)+} \subseteq A^{(\alpha,\gamma,\beta)+}$ and $(A \cap B)^{(\alpha,\gamma,\beta)+} \subseteq B^{(\alpha,\gamma,\beta)+}$

$\Rightarrow (A \cap B)^{(\alpha,\gamma,\beta)+} \subseteq A^{(\alpha,\gamma,\beta)+} \cap B^{(\alpha,\gamma,\beta)+}$

Again, let $\theta \in A^{(\alpha,\gamma,\beta)+} \cap B^{(\alpha,\gamma,\beta)+}$

$\Rightarrow \theta \in A^{(\alpha,\gamma,\beta)+}$ and $\theta \in B^{(\alpha,\gamma,\beta)+}$

$\xi_A(\theta) > \alpha, \xi_B(\theta) > \alpha \Rightarrow min\{\xi_A(\theta), \xi_B(\theta)\} > \alpha$

$\tau_A(\theta) > \gamma, \tau_B(\theta) > \gamma \Rightarrow min\{\tau_A(\theta), \tau_B(\theta)\} > \gamma$

$\zeta_A(\theta) < \beta, \zeta_B(\theta) < \beta \Rightarrow max\{\zeta_A(\theta), \zeta_B(\theta)\} < \beta$

$\Rightarrow \theta \in (A \cap B)^{(\alpha,\gamma,\beta)+}$

$\Rightarrow A^{(\alpha,\gamma,\beta)+} \cap B^{(\alpha,\gamma,\beta)+} \subseteq (A \cap B)^{(\alpha,\gamma,\beta)+}$

Hence, $(A \cap B)^{(\alpha,\gamma,\beta)+} = A^{(\alpha,\gamma,\beta)+} \cap B^{(\alpha,\gamma,\beta)+}$

Theorem 3.7: Let, $A, B \in PFS(U)$, then

(1). $(A \cup B)^{(\alpha,\gamma,\beta)} \supseteq A^{(\alpha,\gamma,\beta)} \cup B^{(\alpha,\gamma,\beta)}$
(2). $(A \cup B)^{(\alpha,\gamma,\beta)+} \supseteq A^{(\alpha,\gamma,\beta)+} \cup B^{(\alpha,\gamma,\beta)+}$

Proof: (1). Since $A \subseteq A \cup B$ and $\subseteq A \cup B$, then from Theorem 3.4, we have

$A^{(\alpha,\gamma,\beta)} \subseteq (A \cup B)^{(\alpha,\gamma,\beta)}$ and $B^{(\alpha,\gamma,\beta)} \subseteq (A \cup B)^{(\alpha,\gamma,\beta)}$

$\Rightarrow A^{(\alpha,\gamma,\beta)} \cup B^{(\alpha,\gamma,\beta)} \subseteq (A \cup B)^{(\alpha,\gamma,\beta)}$

Thus, $(A \cup B)^{(\alpha,\gamma,\beta)} \supseteq A^{(\alpha,\gamma,\beta)} \cup B^{(\alpha,\gamma,\beta)}$.

(2). Since $A \subseteq A \cup B$ and $\subseteq A \cup B$, then from Theorem 3.4, we have

$A^{(\alpha,\gamma,\beta)+} \subseteq (A \cup B)^{(\alpha,\gamma,\beta)+}$ and $B^{(\alpha,\gamma,\beta)+} \subseteq (A \cup B)^{(\alpha,\gamma,\beta)+}$

$\Rightarrow A^{(\alpha,\gamma,\beta)+} \cup B^{(\alpha,\gamma,\beta)+} \subseteq (A \cup B)^{(\alpha,\gamma,\beta)+}$

Thus, $(A \cup B)^{(\alpha,\gamma,\beta)+} \supseteq A^{(\alpha,\gamma,\beta)+} \cup B^{(\alpha,\gamma,\beta)+}$

Theorem 3.8: Let, $A \in PFS(U)$.

(1). If $\alpha_1 \geq \alpha_2, \gamma_1 \geq \gamma_2, \beta_1 \leq \beta_2$, then $A^{(\alpha_1,\gamma_1,\beta_1)} \subseteq A^{(\alpha_2,\gamma_2,\beta_2)}$,
(2). If $\alpha_1 \geq \alpha_2, \gamma_1 \geq \gamma_2, \beta_1 \leq \beta_2$, then $A^{(\alpha_1,\gamma_1,\beta_1)+} \subseteq A^{(\alpha_2,\gamma_2,\beta_2)+}$.

Proof: (1). Let, $\theta \in A^{(\alpha_1,\gamma_1,\beta_1)}$, then $\xi_A(\theta) \geq \alpha_1, \tau_A(\theta) \geq \gamma_1, \zeta_A(\theta) \leq \beta_1$

Since $\alpha_1 \geq \alpha_2, \gamma_1 \geq \gamma_2, \beta_1 \leq \beta_2$, so

$\xi_A(\theta) \geq \alpha_2, \tau_A(\theta) \geq \gamma_2, \zeta_A(\theta) \leq \beta_2$

$\Rightarrow \theta \in A^{(\alpha_2,\gamma_2,\beta_2)}$.

$\therefore A^{(\alpha_1,\gamma_1,\beta_1)} \subseteq A^{(\alpha_2,\gamma_2,\beta_2)}$.

(2). Let, $\theta \in A^{(\alpha_1,\gamma_1,\beta_1)+}$, then $\xi_A(\theta) > \alpha_1, \tau_A(\theta) > \gamma_1, \zeta_A(\theta) < \beta_1$

Since $\alpha_1 \geq \alpha_2, \gamma_1 \geq \gamma_2, \beta_1 \leq \beta_2$, so

$\xi_A(\theta) > \alpha_2, \tau_A(\theta) > \gamma_2, \zeta_A(\theta) < \beta_2$

$\Rightarrow \theta \in A^{(\alpha_2,\gamma_2,\beta_2)+}$.

$\therefore A^{(\alpha_1,\gamma_1,\beta_1)+} \subseteq A^{(\alpha_2,\gamma_2,\beta_2)+}$.

4. Upper Quasi-Cut(Q −Cut) Set for Picture Fuzzy Sets

Definition 4.1: Let, $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)): \theta \in U\}$ be a PFS on U and $\alpha, \gamma, \beta \in [0,1]$, then the upper quasi-cut(Q −cut) of A denoted by $A^{[\alpha,\gamma,\beta]}$ and is defined by

$$A^{[\alpha,\gamma,\beta]} = \{\theta \in U : \alpha + \xi_A(\theta) \geq 1, \gamma + \tau_A(\theta) \geq 1, \beta + \zeta_A(\theta) \leq 1\}.$$

Example 4.1: Let, $U = \{a, b, c, d\}$ and $A = \{(a, 0.3, 0.4, 0.3), (b, 0.4, 0.5, 0.1), (c, 0.6, 0.1, 0.3), (d, 0.5, 0.3, 0.2)\}$ be a picture fuzzy set in U. Let $\alpha = 0.7$, $\gamma = 0.8$, $\beta = 0.2$, then

$$A^{[\alpha,\gamma,\beta]} = \{\theta \in U : 0.7 + \xi_A(\theta) \geq 1, 0.8 + \tau_A(\theta) \geq 1, 0.2 + \zeta_A(\theta) \leq 1\} = \{a, b, d\}.$$

Definition 4.2: Let, $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)): \theta \in U\}$ be a PFS on U and $\alpha, \gamma, \beta \in [0,1]$, then the strong upper quasi-cut(Q −cut) of A denoted by $A^{[\alpha,\gamma,\beta]+}$ and is defined by

$$A^{[\alpha,\gamma,\beta]+} = \{\theta \in U : \alpha + \xi_A(\theta) > 1, \gamma + \tau_A(\theta) > 1, \beta + \zeta_A(\theta) < 1\}$$

Example 4.2: Let, $U = \{a, b, c, d\}$ and $A = \{(a, 0.3, 0.4, 0.3), (b, 0.4, 0.5, 0.1), (c, 0.6, 0.1, 0.3), (d, 0.5, 0.3, 0.2)\}$ be a PFS in U. Let, $\alpha = 0.7$, $\gamma = 0.8$, $\beta = 0.2$, then

$$A^{[\alpha,\gamma,\beta]+} = \{\theta \in U : 0.7 + \xi_A(\theta) > 1, 0.8 + \tau_A(\theta) > 1, 0.2 + \zeta_A(\theta) < 1\} = \{b, d\}.$$

Theorem 4.3: Let, $A, B \in PFS(U)$. If $A \subseteq B$, then

(1). $A^{[\alpha,\gamma,\beta]} \subseteq B^{[\alpha,\gamma,\beta]}$,
(2). $A^{[\alpha,\gamma,\beta]+} \subseteq B^{[\alpha,\gamma,\beta]+}$.

Proof: (1). Let, $\theta \in A^{[\alpha,\gamma,\beta]}$. Then $\alpha + \xi_A(\theta) \geq 1, \gamma + \tau_A(\theta) \geq 1, \beta + \zeta_A(\theta) \leq 1$

As $A \subseteq B$, we have

$\xi_A(\theta) \leq \xi_B(\theta), \tau_A(\theta) \leq \tau_B(\theta), \zeta_A(\theta) \geq \zeta_B(\theta)$

$\Rightarrow \alpha + \xi_B(\theta) \geq 1, \gamma + \tau_B(\theta) \geq 1, \beta + \zeta_B(\theta) \leq 1$

$\Rightarrow \theta \in B^{[\alpha,\gamma,\beta]}$.

$\therefore A^{[\alpha,\gamma,\beta]} \subseteq B^{[\alpha,\gamma,\beta]}$

(2). Let, $\theta \in A^{[\alpha,\gamma,\beta]+}$. Then, $\alpha + \xi_A(\theta) > 1, \gamma + \tau_A(\theta) > 1, \beta + \zeta_A(\theta) < 1$

As $A \subseteq B$, we have

$\mu_B(x) \geq \mu_A(x), \tau_B(\theta) \geq \tau_A(\theta), \zeta_B(\theta) \leq \zeta_A(\theta)$

$\Rightarrow \alpha + \xi_B(\theta) > 1, \gamma + \tau_B(\theta) > 1, \beta + \zeta_B(\theta) < 1$

$\Rightarrow \theta \in B^{[\alpha,\gamma,\beta]}$.

$\therefore A^{[\alpha,\gamma,\beta]+} \subseteq B^{[\alpha,\gamma,\beta]+}$

Theorem 4.4: Let, $A, B \in PFS(U)$. If $A = B$, then

(1). $A^{[\alpha,\gamma,\beta]} = B^{[\alpha,\gamma,\beta]}$,
(2). $A^{[\alpha,\gamma,\beta]+} = B^{[\alpha,\gamma,\beta]+}$.

Proof: (1). Let, $\theta \in A^{[\alpha,\gamma,\beta]}$, then $\alpha + \xi_A(\theta) \geq 1, \gamma + \tau_A(\theta) \geq 1, \beta + \zeta_A(\theta) \leq 1$

As $A = B$, we have

$\xi_A(\theta) = \xi_B(\theta), \tau_A(\theta) = \tau_B(\theta), \zeta_A(\theta) = \zeta_B(\theta)$

$\Rightarrow \alpha + \xi_B(\theta) \geq 1, \gamma + \tau_B(\theta) \geq 1, \beta + \zeta_B(\theta) \leq 1$

$\Rightarrow \theta \in B^{[\alpha,\gamma,\beta]}$.

$\therefore A^{[\alpha,\gamma,\beta]} \subseteq B^{[\alpha,\gamma,\beta]}$.

Again, let $\theta \in B^{[\alpha,\gamma,\beta]}$, then $\alpha + \xi_B(\theta) \geq 1, \gamma + \tau_B(\theta) \geq 1, \beta + \zeta_B(\theta) \leq 1$.

As $A = B$, we have

$\xi_A(\theta) = \xi_B(\theta), \tau_A(\theta) = \tau_B(\theta), \zeta_A(\theta) = \zeta_B(\theta)$

$\Rightarrow \alpha + \xi_A(\theta) \geq 1, \gamma + \tau_A(\theta) \geq 1, \beta + \zeta_A(\theta) \leq 1$

$\Rightarrow \theta \in A^{[\alpha,\gamma,\beta]}$.

$\therefore B^{[\alpha,\gamma,\beta]} \subseteq A^{[\alpha,\gamma,\beta]}$.

Thus, $A^{[\alpha,\gamma,\beta]} = B^{[\alpha,\gamma,\beta]}$.

(2). Let, $\theta \in A^{[\alpha,\gamma,\beta]+}$, then $\alpha + \xi_A(\theta) > 1, \gamma + \tau_A(\theta) > 1, \beta + \zeta_A(\theta) < 1$

As $A = B$, we have

$\xi_A(\theta) = \xi_B(\theta), \tau_A(\theta) = \tau_B(\theta), \zeta_A(\theta) = \zeta_B(\theta)$

$\Rightarrow \alpha + \xi_B(\theta) > 1, \gamma + \tau_B(\theta) > 1, \beta + \zeta_B(\theta) < 1$

$\Rightarrow \theta \in B^{[\alpha,\gamma,\beta]+}$.

$\therefore A^{[\alpha,\gamma,\beta]+} \subseteq B^{[\alpha,\gamma,\beta]+}$.

Again, let $\theta \in B^{[\alpha,\gamma,\beta]+}$, then $\alpha + \xi_B(\theta) > 1, \gamma + \tau_B(\theta) > 1, \beta + \zeta_B(\theta) < 1$.

As $A = B$, we have

$\xi_A(\theta) = \xi_B(\theta), \tau_A(\theta) = \tau_B(\theta), \zeta_A(\theta) = \zeta_B(\theta)$

$\Rightarrow \alpha + \xi_A(\theta) > 1, \gamma + \tau_A(\theta) > 1, \beta + \zeta_A(\theta) < 1$

$\Rightarrow \theta \in A^{[\alpha,\gamma,\beta]+}$.

$\therefore B^{[\alpha,\gamma,\beta]+} \subseteq A^{[\alpha,\gamma,\beta]+}$.

Thus, $A^{[\alpha,\gamma,\beta]+} = B^{[\alpha,\gamma,\beta]+}$.

Theorem 4.5: Let, $A, B \in PFS(U)$, then

(1). $(A \cap B)^{[\alpha,\gamma,\beta]} = A^{[\alpha,\gamma,\beta]} \cap B^{[\alpha,\gamma,\beta]}$,
(2). $(A \cap B)^{[\alpha,\gamma,\beta]+} = A^{[\alpha,\gamma,\beta]+} \cap B^{[\alpha,\gamma,\beta]+}$.

Proof: (1). We have,

$$A \cap B \subseteq A \text{ and } A \cap B \subseteq B$$

Hence from Theorem 4.3, we have

$(A \cap B)^{[\alpha,\gamma,\beta]} \subseteq A^{[\alpha,\gamma,\beta]}$ and $(A \cap B)^{[\alpha,\gamma,\beta]} \subseteq B^{[\alpha,\gamma,\beta]}$

$\Rightarrow (A \cap B)^{[\alpha,\gamma,\beta]} \subseteq A^{[\alpha,\gamma,\beta]} \cap B^{[\alpha,\gamma,\beta]}$

Again, let $\theta \in A^{[\alpha,\gamma,\beta]} \cap B^{[\alpha,\gamma,\beta]}$

$\Rightarrow \theta \in A^{[\alpha,\gamma,\beta]}$ and $\theta \in B^{[\alpha,\gamma,\beta]}$

$\alpha + \xi_A(\theta) \geq 1, \alpha + \xi_B(\theta) \geq 1$
$\Rightarrow min\{\xi_A(\theta), \xi_B(\theta)\} + \alpha \geq 1$

$\gamma + \tau_A(\theta) \geq 1, \gamma + \tau_B(\theta) \geq 1$
$\Rightarrow min\{\tau_A(\theta), \tau_B(\theta)\} + \gamma \geq 1$

$\beta + \zeta_A(\theta) \leq 1, \beta + \zeta_B(\theta) \leq 1$
$\Rightarrow max\{\zeta_A(\theta), \zeta_B(\theta)\} + \beta \leq 1$

$\Rightarrow \theta \in (A \cap B)^{[\alpha,\gamma,\beta]}$

$\Rightarrow A^{[\alpha,\gamma,\beta]} \cap B^{[\alpha,\gamma,\beta]} \subseteq (A \cap B)^{[\alpha,\gamma,\beta]}$

Hence, $(A \cap B)^{[\alpha,\gamma,\beta]} = A^{[\alpha,\gamma,\beta]} \cap B^{[\alpha,\gamma,\beta]}$

(2). We have, $A \cap B \subseteq A$ and $A \cap B \subseteq B$

Hence from Theorem 4.3, we have

$(A \cap B)^{[\alpha,\gamma,\beta]+} \subseteq A^{[\alpha,\gamma,\beta]+}$ and $(A \cap B)^{[\alpha,\gamma,\beta]+} \subseteq B^{[\alpha,\gamma,\beta]+}$

$\Rightarrow (A \cap B)^{[\alpha,\gamma,\beta]+} \subseteq A^{[\alpha,\gamma,\beta]+} \cap B^{[\alpha,\gamma,\beta]+}$

Again, let $\theta \in A^{[\alpha,\gamma,\beta]+} \cap B^{[\alpha,\gamma,\beta]+}$

$\Rightarrow \theta \in A^{[\alpha,\gamma,\beta]+}$ and $\theta \in B^{[\alpha,\gamma,\beta]+}$

$\alpha + \xi_A(\theta) > 1$, $\alpha + \xi_B(\theta) > 1$
$\Rightarrow min\{\xi_A(\theta), \xi_B(\theta)\} + \alpha > 1$

$\gamma + \tau_A(\theta) > 1, \gamma + \tau_B(\theta) > 1$
$\Rightarrow min\{\tau_A(\theta), \tau_B(\theta)\} + \gamma > 1$

$\beta + \zeta_A(\theta) < 1, \beta + \zeta_B(\theta) < 1$
$\Rightarrow max\{\zeta_A(\theta), \zeta_B(\theta)\} + \beta < 1$

$\Rightarrow \theta \in (A \cap B)^{[\alpha,\gamma,\beta]+}$

$\Rightarrow A^{[\alpha,\gamma,\beta]+} \cap B^{[\alpha,\gamma,\beta]+} \subseteq (A \cap B)^{[\alpha,\gamma,\beta]+}$

Hence, $(A \cap B)^{[\alpha,\gamma,\beta]+} = A^{[\alpha,\gamma,\beta]+} \cap B^{[\alpha,\gamma,\beta]+}$

Theorem 4.6: Let $A, B \in PFS(X)$, then

(1). $(A \cup B)^{[\alpha,\gamma,\beta]} \supseteq A^{[\alpha,\gamma,\beta]} \cup B^{[\alpha,\gamma,\beta]}$,
(2). $(A \cup B)^{[\alpha,\gamma,\beta]+} \supseteq A^{[\alpha,\gamma,\beta]+} \cup B^{[\alpha,\gamma,\beta]+}$.

Proof: (1). Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then

Hence from Theorem 4.3, we have

$A^{[\alpha,\gamma,\beta]} \subseteq (A \cup B)^{[\alpha,\gamma,\beta]}$ and $B^{[\alpha,\gamma,\beta]} \subseteq (A \cup B)^{[\alpha,\gamma,\beta]}$

$\Rightarrow A^{[\alpha,\gamma,\beta]} \cup B^{[\alpha,\gamma,\beta]} \subseteq (A \cup B)^{[\alpha,\gamma,\beta]}$

Thus, $(A \cup B)^{[\alpha,\gamma,\beta]} \supseteq A^{[\alpha,\gamma,\beta]} \cup B^{[\alpha,\gamma,\beta]}$

(2). Since $A \subseteq A \cup B$ and $\subseteq A \cup B$, then

Hence from Theorem 4.3, we have

$A^{[\alpha,\gamma,\beta]+} \subseteq (A \cup B)^{[\alpha,\gamma,\beta]+}$ and $B^{[\alpha,\gamma,\beta]+} \subseteq (A \cup B)^{[\alpha,\gamma,\beta]+}$

$\Rightarrow A^{[\alpha,\gamma,\beta]+} \cup B^{[\alpha,\gamma,\beta]+} \subseteq (A \cup B)^{[\alpha,\gamma,\beta]+}$

Thus, $(A \cup B)^{[\alpha,\gamma,\beta]+} \supseteq A^{[\alpha,\gamma,\beta]+} \cup B^{[\alpha,\gamma,\beta]+}$

Theorem 4.7: Let, $A \in PFS(U)$.

(1). If $\alpha_1 \leq \alpha_2$, $\gamma_1 \leq \gamma_2$, $\beta_1 \geq \beta_2$, then $A^{[\alpha_1,\gamma_1,\beta_1]} \subseteq A^{[\alpha_2,\gamma_2,\beta_2]}$,
(2). If $\alpha_1 \leq \alpha_2$, $\gamma_1 \leq \gamma_2$, $\beta_1 \geq \beta_2$, then $A^{[\alpha_1,\gamma_1,\beta_1]+} \subseteq A^{[\alpha_2,\gamma_2,\beta_2]+}$.

Proof: (1). Let, $\theta \in A^{[\alpha_1,\gamma_1,\beta_1]}$, then $\alpha_1 + \xi_A(\theta) \geq 1$, $\gamma_1 + \tau_A(\theta) \geq 1$, $\beta_1 + \zeta_A(\theta) \leq 1$

Since $\alpha_1 \leq \alpha_2$, $\gamma_1 \leq \gamma_2$, $\beta_1 \geq \beta_2$

$\Rightarrow \alpha_2 + \xi_A(\theta) \geq 1$, $\gamma_2 + \tau_A(\theta) \geq 1$, $\beta_2 + \zeta_A(\theta) \leq 1$

$\Rightarrow \theta \in A^{[\alpha_2,\gamma_2,\beta_2]}$.

$\therefore A^{[\alpha_1,\gamma_1,\beta_1]} \subseteq A^{[\alpha_2,\gamma_2,\beta_2]}$.

(2). Let, $\theta \in A^{[\alpha_1,\gamma_1,\beta_1]+}$, then $\alpha_1 + \xi_A(\theta) > 1$, $\gamma_1 + \tau_A(\theta) > 1$, $\beta_1 + \zeta_A(\theta) < 1$

Since $\alpha_1 \leq \alpha_2$, $\gamma_1 \leq \gamma_2$, $\beta_1 \geq \beta_2$

$\Rightarrow \alpha_2 + \xi_A(\theta) > 1$, $\gamma_2 + \tau_A(\theta) > 1$, $\beta_2 + \zeta_A(\theta) < 1$

$\Rightarrow \theta \in A^{[\alpha_2,\gamma_2,\beta_2]+}$.

$\therefore A^{[\alpha_1,\gamma_1,\beta_1]+} \subseteq A^{[\alpha_2,\gamma_2,\beta_2]+}$.

5. Lower Quasi-Cut(Q −Cut) Set for Picture Fuzzy Sets

Definition 5.1: Let, $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U and $\alpha, \gamma, \beta \in [0,1]$, then the lower quasi-cut(Q −cut) of A denoted by $A_{[\alpha,\gamma,\beta]}$ and is defined by

$$A_{[\alpha,\gamma,\beta]} = \{\theta \in U : \alpha + \xi_A(\theta) \leq 1 , \gamma + \tau_A(\theta) \leq 1 , \beta + \zeta_A(\theta) \geq 1\}.$$

Example 5.1: Let, $U = \{a, b, c, d, e\}$ and $A = \left\{ \begin{matrix} (a, 0.4, 0.4, 0.2), (b, 0.2, 0.6, 0.1), (c, 0.4, 0.1, 0.5), (d, 0.3, 0.2, 0.3), \\ (e, 0.2, 0.3, 0.4) \end{matrix} \right\}$ be a picture fuzzy set in U.

Let $\alpha = 0.6$, $\gamma = 0.2$, $\beta = 0.8$, then

$A_{[\alpha,\gamma,\beta]} = \{\theta \in U : 0.6 + \xi_A(\theta) \leq 1 , 0.2 + \tau_A(\theta) \leq 1 , 0.8 + \zeta_A(\theta) \geq 1\} = \{a, c, d, e\}$.

Definition 5.2: Let, $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U and $\alpha, \gamma, \beta \in [0,1]$, then the strong lower quasi-cut(Q −cut) of A denoted by $A_{[\alpha,\gamma,\beta]+}$ and is defined by

$$A_{[\alpha,\gamma,\beta]+} = \{\theta \in U : \alpha + \xi_A(\theta) < 1 , \gamma + \tau_A(\theta) < 1 , \beta + \zeta_A(\theta) > 1\}.$$

Example 5.2: Let, $U = \{a, b, c, d, e\}$ and $A = \{(a, 0.4, 0.4, 0.2), (b, 0.2, 0.6, 0.1), (c, 0.4, 0.1, 0.5), (d, 0.3, 0.2, 0.3), (e, 0.2, 0.3, 0.4)\}$ be a PFS in U. Let, $\alpha = 0.6,\ \gamma = 0.2,\ \beta = 0.8$, then

$A_{[\alpha,\gamma,\beta]+} = \{\theta \in U: 0.6 + \xi_A(\theta) < 1 , 0.2 + \tau_A(\theta) < 1 , 0.8 + \zeta_A(\theta) > 1\} = \{d, e\}$.

It can be noted that, $A_{[\alpha,\gamma,\beta]+} \subseteq A_{[\alpha,\gamma,\beta]}$.

Theorem 5.3: Let, $A, B \in PFS(U)$. If $A \subseteq B$, then

(1). $B_{[\alpha,\gamma,\beta]} \subseteq A_{[\alpha,\gamma,\beta]}$,
(2). $B_{[\alpha,\gamma,\beta]+} \subseteq A_{[\alpha,\gamma,\beta]+}$.

Proof: (1). Let, $\theta \in B_{[\alpha,\gamma,\beta]}$. Then $\alpha + \xi_B(\theta) \leq 1 , \gamma + \tau_B(\theta) \leq 1 , \beta + \zeta_B(\theta) \geq 1$

As $A \subseteq B$, we have

$\xi_A(\theta) \leq \xi_B(\theta), \tau_A(\theta) \leq \tau_B(\theta) , \zeta_A(\theta) \geq \zeta_B(\theta)$

$\Rightarrow \alpha + \xi_A(\theta) \leq 1 , \gamma + \tau_A(\theta) \leq 1 , \beta + \zeta_A(\theta) \geq 1$

$\Rightarrow \theta \in A^{[\alpha,\gamma,\beta]}$.

$\therefore\ B^{[\alpha,\gamma,\beta]} \subseteq A^{[\alpha,\gamma,\beta]}$.

(2). Let, $\theta \in B_{[\alpha,\gamma,\beta]+}$. Then $\alpha + \xi_B(\theta) < 1 , \gamma + \tau_B(\theta) < 1 , \beta + \zeta_B(\theta) > 1$

As $A \subseteq B$, we have

$\xi_A(\theta) \leq \xi_B(\theta), \tau_A(\theta) \leq \tau_B(\theta) , \zeta_A(\theta) \geq \zeta_B(\theta)$

$\Rightarrow\ \alpha + \xi_A(\theta) < 1 , \gamma + \tau_A(\theta) < 1 , \beta + \zeta_A(\theta) > 1$

$\Rightarrow \theta \in A^{[\alpha,\gamma,\beta]+}$.

$\therefore\ B^{[\alpha,\gamma,\beta]+} \subseteq A^{[\alpha,\gamma,\beta]+}$

Theorem 5.4: Let, $A, B \in PFS(U)$, then $A = B$ implies

(1). $A_{[\alpha,\gamma,\beta]} = B_{[\alpha,\gamma,\beta]}$,
(2). $A_{[\alpha,\gamma,\beta]+} = B_{[\alpha,\gamma,\beta]+}$.

Proof: (1). Let, $\theta \in A_{[\alpha,\gamma,\beta]}$, then $\alpha + \xi_A(\theta) \leq 1 , \gamma + \tau_A(\theta) \leq 1 , \beta + \zeta_A(\theta) \geq 1$

As $A = B$, we have

$\xi_A(\theta) = \xi_B(\theta), \tau_A(\theta) = \tau_B(\theta) , \zeta_A(\theta) = \zeta_B(\theta)$

$\Rightarrow\ \alpha + \xi_B(\theta) \leq 1 , \gamma + \tau_B(\theta) \leq 1 , \beta + \zeta_B(\theta) \geq 1$

$\Rightarrow\ \theta \in B_{[\alpha,\gamma,\beta]}$.

$\therefore\ A_{[\alpha,\gamma,\beta]} \subseteq B_{[\alpha,\gamma,\beta]}$.

Again, let $\theta \in B_{[\alpha,\gamma,\beta]}$, then $\alpha + \xi_B(\theta) \leq 1 , \gamma + \tau_B(\theta) \leq 1 , \beta + \zeta_B(\theta) \geq 1$.

As $A = B$, we have

$\xi_A(\theta) = \xi_B(\theta), \tau_A(\theta) = \tau_B(\theta) , \zeta_A(\theta) = \zeta_B(\theta)$

$\Rightarrow\ \alpha + \xi_A(\theta) \leq 1 , \gamma + \tau_A(\theta) \leq 1 , \beta + \zeta_A(\theta) \geq 1$

$\Rightarrow \theta \in A_{[\alpha,\gamma,\beta]}$.

$\therefore\ B_{[\alpha,\gamma,\beta]} \subseteq A_{[\alpha,\gamma,\beta]}$.

Thus, $A_{[\alpha,\gamma,\beta]} = B_{[\alpha,\gamma,\beta]}$.

(2). Let, $\theta \in A_{[\alpha,\gamma,\beta]+}$, then $\alpha + \xi_A(\theta) < 1 , \gamma + \tau_A(\theta) > 1 , \beta + \zeta_A(\theta) > 1$

As $A = B$, we have

$\xi_A(\theta) = \xi_B(\theta), \tau_A(\theta) = \tau_B(\theta) , \zeta_A(\theta) = \zeta_B(\theta)$

$\Rightarrow\ \alpha + \xi_B(\theta) < 1 , \gamma + \tau_B(\theta) > 1 , \beta + \zeta_B(\theta) > 1$

$\Rightarrow\ \theta \in B_{[\alpha,\gamma,\beta]+}$.

$\therefore\ A_{[\alpha,\gamma,\beta]+} \subseteq B_{[\alpha,\gamma,\beta]+}$.

Again,

Let, $\theta \in B_{[\alpha,\gamma,\beta]+}$, then $\alpha + \xi_B(\theta) < 1 , \gamma + \tau_B(\theta) > 1 , \beta + \zeta_B(\theta) > 1$.

As $A = B$, we have

$\xi_A(\theta) = \xi_B(\theta), \tau_A(\theta) = \tau_B(\theta) , \zeta_A(\theta) = \zeta_B(\theta)$

$\Rightarrow\ \alpha + \xi_A(\theta) < 1 , \gamma + \tau_A(\theta) > 1 , \beta + \zeta_A(\theta) > 1$

$\Rightarrow\ \theta \in A_{[\alpha,\gamma,\beta]+}$.

$\therefore\ B_{[\alpha,\gamma,\beta]+} \subseteq A_{[\alpha,\gamma,\beta]+}$.

Thus, $A_{[\alpha,\gamma,\beta]+} = B_{[\alpha,\gamma,\beta]+}$.

Theorem 5.5: Let, $A, B \in PFS(U)$, then

(1). $(A \cup B)_{[\alpha,\gamma,\beta]} = A_{[\alpha,\gamma,\beta]} \cap B_{[\alpha,\gamma,\beta]}$,
(2). $(A \cup B)_{[\alpha,\gamma,\beta]+} = A_{[\alpha,\gamma,\beta]+} \cap B_{[\alpha,\gamma,\beta]+}$,
(3). $(A \cup B)_{[\alpha,\gamma,\beta]} \subseteq A_{[\alpha,\gamma,\beta]} \cup B_{[\alpha,\gamma,\beta]}$,
(4). $(A \cup B)_{[\alpha,\gamma,\beta]+} \subseteq A_{(\alpha,\gamma,\beta)+} \cup B_{[\alpha,\gamma,\beta]+}$.

Proof: (1). Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so from the Theorem 5.3, we have,

$(A \cup B)_{[\alpha,\gamma,\beta]} \subseteq A_{[\alpha,\gamma,\beta]}$ and $(A \cup B)_{[\alpha,\gamma,\beta]} \subseteq B_{[\alpha,\gamma,\beta]}$

$\Rightarrow (A \cup B)_{[\alpha,\gamma,\beta]} \subseteq A_{[\alpha,\gamma,\beta]} \cap B_{[\alpha,\gamma,\beta]}$.

Again, let $\theta \in A_{[\alpha,\gamma,\beta]} \cap B_{[\alpha,\gamma,\beta]}$

$\Rightarrow x \in A_{[\alpha,\gamma,\beta]}$ and $x \in B_{[\alpha,\gamma,\beta]}$

$\Rightarrow \alpha + \xi_A(\theta) \leq 1 , \alpha + \xi_B(\theta) \leq 1$
$\Rightarrow max\{\xi_A(\theta), \xi_B(\theta)\} + \alpha \leq 1$

$\gamma + \tau_A(\theta) \leq 1, \gamma + \tau_B(\theta) \leq 1$
$\Rightarrow min\{\tau_A(\theta), \tau_B(\theta)\} + \gamma \leq 1$

$\beta + \zeta_A(\theta) \geq 1, \beta + \zeta_B(\theta) \geq 1 \Rightarrow min\{\zeta_A(\theta), \zeta_B(\theta)\} + \beta \geq 1$

$\Rightarrow \theta \in (A \cup B)_{[\alpha,\gamma,\beta]}$.

Therefore, $A_{[\alpha,\gamma,\beta]} \cap B_{[\alpha,\gamma,\beta]} \subseteq (A \cup B)_{[\alpha,\gamma,\beta]}$.

Hence, $(A \cup B)_{[\alpha,\gamma,\beta]} = A_{[\alpha,\gamma,\beta]} \cap B_{[\alpha,\gamma,\beta]}$.

(2). Proof is similar to proof of (1).

(3). Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so from the Theorem 5.3, we have,

$(A \cup B)_{[\alpha,\gamma,\beta]} \subseteq A_{[\alpha,\gamma,\beta]}$ and $(A \cup B)_{[\alpha,\gamma,\beta]} \subseteq B_{[\alpha,\gamma,\beta]}$

$\Rightarrow (A \cup B)_{[\alpha,\gamma,\beta]} \subseteq A_{[\alpha,\gamma,\beta]} \cup B_{[\alpha,\gamma,\beta]}$.

(4). Proof is similar to proof of (3).

Theorem 5.6: Let, $A \in PFS(U)$.

(1). If $\alpha_1 \leq \alpha_2 , \gamma_1 \leq \gamma_2 , \beta_1 \geq \beta_2$, then $A_{[\alpha_2,\gamma_2,\beta_2]} \subseteq A_{[\alpha_1,\gamma_1,\beta_1]}$,

(2). If $\alpha_1 \leq \alpha_2 , \gamma_1 \leq \gamma_2 , \beta_1 \geq \beta_2$, then $A_{[\alpha_2,\gamma_2,\beta_2]+} \subseteq A_{[\alpha_1,\gamma_1,\beta_1]+}$.

Proof: (1). Let, $\theta \in A_{[\alpha_2,\gamma_2,\beta_2]}$, then

$\alpha_2 + \xi_A(\theta) \leq 1 , \gamma_2 + \tau_A(\theta) \leq 1 , \beta_2 + \zeta_A(\theta) \geq 1$

Since $\alpha_1 \leq \alpha_2 , \gamma_1 \leq \gamma_2 , \beta_1 \geq \beta_2$

$\Rightarrow \alpha_1 + \xi_A(\theta) \leq 1 , \gamma_1 + \tau_A(\theta) \leq 1 , \beta_1 + \zeta_A(\theta) \geq 1$

$\Rightarrow \theta \in A_{[\alpha_1,\gamma_1,\beta_1]}$.

$\therefore A_{[\alpha_2,\gamma_2,\beta_2]} \subseteq A_{[\alpha_1,\gamma_1,\beta_1]}$.

(2). Let, $\theta \in A_{[\alpha_2,\gamma_2,\beta_2]+}$, then $\alpha_2 + \xi_A(\theta) < 1 , \gamma_2 + \tau_A(\theta) < 1 , \beta_2 + \zeta_A(\theta) > 1$

Since $\alpha_1 \leq \alpha_2 , \gamma_1 \leq \gamma_2 , \beta_1 \geq \beta_2$

$\Rightarrow \alpha_1 + \xi_A(\theta) \leq 1 , \gamma_1 + \tau_A(\theta) \leq 1 , \beta_1 + \zeta_A(\theta) \geq 1$

$\Rightarrow \theta \in A_{[\alpha_1,\gamma_1,\beta_1]+}$.

$\therefore A_{[\alpha_2,\gamma_2,\beta_2]+} \subseteq A_{[\alpha_1,\gamma_1,\beta_1]+}$.

6. Conclusions

Consider a picture fuzzy set, which serves as a direct extension of the intuitionistic fuzzy set and is a recently developed tool for addressing uncertainty. It effectively models situations where multiple responses are possible, such as yes, abstain, and no. The cut sets of picture fuzzy sets play a crucial role in arithmetic operations, including addition, subtraction, multiplication, and division of picture fuzzy numbers. This paper discusses the upper (α, δ, β) −cut set for picture fuzzy sets, highlighting various properties. Additionally, the definitions of upper and lower quasi cut sets for picture fuzzy sets are provided, along with a comprehensive description of several related properties. The concepts of cut sets will be utilized to formulate decomposition theorems and will find applications in diverse areas such as picture fuzzy analysis, picture fuzzy algebra, picture fuzzy reasoning, and picture fuzzy clustering, particularly in decision-making processes.

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