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Modeling Stiff Differential Equations: A COVID-19 Case Study in Verification and Application

Research Article

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ABSTRACT

In this research, we explore the application of the Differential Transform Method (DTM) and Local Differential Transform Method (LDTM) to solve diverse types of nonlinear differential equations encountered in stiff systems. Stiff differential equations represent a category of mathematical models arising in scientific and engineering contexts where the solution displays both rapid and gradual changes on distinct time scales. The numerical solution of these systems can be challenging due to the broad range of time scales involved, leading to potential issues of numerical instability and accuracy. The exploration of stiff differential equations remains an active research area, with continuous efforts focused on devising novel numerical methods to enhance the accuracy, efficiency, and robustness of solving these intricate systems. Various types of initial value problems are considered comparatively, examining the stability of the DTM and LDTM methods across different time intervals. To effectively characterize the nonlinear behavior of the studied problems, we employ the Duffing model, which represents a forced oscillator coupled with a spring exerting a rebalancing force. Within this investigation, a novel numerical estimation method is introduced based on the modified LDTM to address the nonlinear cubic Duffing equation, both dissipative and non-dissipative cases.

Keywords: *Stiff System, DTM, LDTM, Duffing Equation, Flip Bifurcation*

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1. Introduction

The term "Stiff" in the context of differential equations refers to a particular type of ordinary differential equation (ODE) that poses numerical challenges for certain solution methods. Stiff ODEs are characterized by having widely varying time scales or rapidly changing behavior in different regions of the solution domain. In simple terms, they are equations that contain both "fast" and "slow" processes, and the stiffness arises from the disparity in their rates of change. The Local Differential Transform Method (LDTM) is a variation of the DTM, which is a mathematical technique used to solve differential equations. The LDTM is used to approximate solutions to differential equations with non-analytic coefficients. Like the DTM, the LDTM involves a series of transforms applied to the original differential equation. However, unlike the DTM, the LDTM applies the transforms locally, in small intervals, rather than globally. This allows for the approximation of solutions to Differential Equations (DEs) with non-analytic coefficients that unsolvable using traditional analytic methods. In [1], Radau IIA methods are widely recognized for their effectiveness in solving stiff differential equations numerically. This paper introduces RADAU, a novel implementation that incorporates a variable-order strategy. It begins with an overview of the historical progression of these approaches, and explores their theoretical characteristics. In [2], the aim is to advocate for the efficient application of SERK methods in solving such problems. Traditionally, 2^{nd} -order finite differences have been utilized as territorial deviations to demonstrate the effectiveness of these frameworks. This study explores the performance of these algorithms when alternative approaches, including those with faster convergence, are employed.

In [3], this work introduces and analyzes a novel splitting approach, termed the Canonical Euler Splitting (CES) method, and designed for the well ordered numerical solution of nonlinear compound stiff cases. It applies to various evolution equations, including ODEs, and semi-discrete unsteady PDEs. The method enhances computational speed while maintaining solution accuracy. In [4], they introduce a variational formulation for ordinary differential equation systems. In [5], this study applies the DTM to solve delay differential equations (DDEs). Utilizing the method of steps for DDEs and leveraging the computer algebra system Mathematica, DTM is effectively used to obtain analytic solutions for various ODEs, including a neural delay DEs. In [6], the DTM is employed to solve initial-value problems. This method, which utilizes a fixed grid size, approximates solutions for both linear and nonlinear initial-value problems. In [7, 8], the authors review significant theories related to these formulas and discuss practical applications where they prove particularly effective. Additionally, they examine an implicit Continuous Block Backward Differentiation Formula (CBBDF) for ODEs. This method generates a block of current values at every step, at the same time providing approximate solutions for the ODEs, where the number of step is specified. In [9], a novel numerical approximation technique utilizing the DTM is proposed for the nonlinear cubic Duffing equation, considering both damped and undamped cases. Given that exact solutions for all initial conditions are unavailable in the literature, an exact solution is first derived for specific parameters using the Kudryashov process to evaluate the accuracy of the proposed technique. In [10], a set of second derivative formulas is developed, and their stability is analyzed. This n step $(n+2)$ and order formulas are shown to be applicable for stiff systems for $n \le 7$. They have been incorporated into a variable order, and variable step method, and various numerical results are provided. In [11], the authors introduce a novel category of 2^{nd}-derivative multistep process, and discuss their stability analysis, highlighting an enhancement in the stability domain. By implementing a straightforward modification, they leverage the ability to solve algebraic equations that share the analogous coefficient matrix at every step. Furthermore, by using the Iterative Over-Relaxation Method (IOM) to address the resulting linear systems, the explicit form of the coefficient matrix is not required. The objective of [12] is to identify a specific class from the extensive range of general linear methods that shows significant potential for effective actualization. This selected sector has various applications depending on the stiffness of the problem being addressed. A specialized second derivative multistep method (SDMM) is derived. In [13], Robertson's example illustrates representative reaction kinetics through a system of three ODEs. Following an introduction to its applications in chemical engineering, a theoretical analysis of stiffness is conducted. The findings are validated through numerical experiments, contrasting the performance of both a non-stiff and a stiff numerical solver. The methods discussed in this study present a viable approach to a problem believed to be stiff. The LDTM has proven to be an effective method for approximating solutions to a wide range of differential equations, including those with discontinuous or non-analytic coefficients. It has also been used to approximate solutions to differential equations with fractional derivatives, which cannot be solved using traditional analytic methods. However, like any numerical method, the exactness of the LDTM depends on the size of the sub domains chosen and the number of iterations performed. The Duffing model, spell out as a forced oscillator with a spring exhibiting a reserving force, describes the nonlinear behavior of various problems. This study introduces a novel numerical approximation technique using the differential transform method to analyze nonlinear DEs related to COVID-19. As exact solutions for all initial conditions are not readily available in the literature, the proposed approach is compared with both the semi-analytic DTM and the 4th -order Runge-Kutta method. While the semi-analytic DTM is effective for short time intervals, the proposed technique demonstrates its ability to capture nonlinear behavior over extended periods. Additionally, results indicate that the present method offers improved accuracy.

2. Exploration of the LDM

The basic principles and essential theorems related to the Differential Transform Method (DTM), along with its applications to various types of DEs, are available. To facilitate understanding, provide an overview of the DTM. The formal definition of the differential transform for the n^{th} derivative of $F(t)$ is written as

$$F(t)=\frac{1}{n}\left[\frac{d^n f(t)}{dt^n}\right]_{t=t_0}. \tag{1}$$

Here, $f(t)$ represents the main function, and $F(n)$ denotes the transformed function. The differential inverse transform of $F(n)$ is well denoted by the expression

$$f(t)=\sum_{n=0}^{\infty} F(n)h^n.\ \ where\, h=t-t_0 \tag{2}$$

By manipulating Equation (1), and Equation (2), we can deduce the subsequent relationship:

$$f(t)=\left(\sum_{n=0}^{\infty}\frac{h^n}{n!}\frac{d^n f(t)}{dt^n}\right)_{t=t_0}. \tag{3}$$

This statement underscores that the origin of the differential transform concept can be traced back to the Taylor series. However, the DTM does not symbolically compute derivatives. Instead, it calculates relative derivatives through an iterative process, as narrated by the transformed equations of the main function. In real life, the $f(t)$ is frequently written as a finite series, and Equation (2) can be expressed as

$$f(t) = \sum_{n=0}^{k} F(n)h^n. \qquad (4)$$

The determination of the value for k relies on the convergence properties of the natural frequency. In the following theorem, we examine the differential transformation of products that involve single-valued functions. These findings play a significant role in our methodology for solving different types of equations.

Table 1: Basic formula of transfer functions.

Function	Transformed Function
$f(t) = f_1(t) \pm f_2(t)$	$F(n) = F_1(n) \pm F_2(n)$
$f(t) = \alpha f_1(t)$	$F(n) = \alpha F_1(n)$
$f(t) = \dfrac{df_1(t)}{dt}$	$F(n) = (n+1)F_1(n+1)$
$f(t) = \dfrac{d^m f_1(t)}{dt^m}$	$F(n) = (n+1)(n+2)\ldots(n+m)F_1(n+m)$
$f(t) = t^m$	$F(\mathrm{n}) = \begin{cases} 1 & if\ n = m \\ C(m,n)t_k^{(n-m)} & if\ n \neq m \end{cases}$
$f(t) = exp(\gamma t)$	$F(n) = exp(\gamma t_n)\dfrac{\gamma^n}{n!}$
$f(t) = f_1(t)\, f_2(t)$	$F(\mathrm{n}) = \sum_{l=0}^{k} F_1(l)F_2(n-l)$
$f(t) = f_1^m(t)$	$F(n) = \sum_{l=0}^{n} F_1^{m-1}F_1(n-l)$

3. Error Analysis

Considering the differential equation

$$f'(t) = F\big(f(t),t\big), t > 0, t(0) = t_0. \qquad (5)$$

Here, $t_0 \in R^m$, and the function $F: R \times R^m \to R^m$. We can express the LDTM solution for Equation (5) as follows:

$$Y_n(j+1) = \frac{1}{j+1}\big[F(n), t_n\big],\ j = 0,1,\ldots,M-1, \qquad (6)$$

In this context, the transformed vector-valued functions are denoted by X_j and F. Utilizing a Taylor expansion, we can articulate the precise solution to Equation (5) at the time t_{j+1}:

$$f(t_{j+1}) = \sum_{j=0}^{\infty} Y_j(j)h^j = \sum_{j=0}^{n} f_j h^j + h\rho_m. \qquad (8)$$

Here, $m = 0,1,2,\ldots,M-1$, and ρ_m represents the local truncation error. This error can be calculated by applying the residual formula derived from the Taylor series

$$\rho_m = Y_m(j+1)h^j = \frac{h^j}{j+1}\left[F(K,t_c)\right)$$
$$= \frac{h^j}{(j+1)!}\left[\frac{d^j G(f(t),t)}{dt^j}\right] \vee t = t, t_c \in (\mathrm{t}_j,\ \mathrm{t}_{j+1}). \quad (9)$$

Hence, this approach furnishes a localized estimate of the precise solution with a specified order of K. To ensure the convergence of the numerical scheme, it is imperative to constrain the global error within the scheme. For a more comprehensive examination of the overall discretization error, let's assume linearity in the problem, denoted as $F(f(t),t) = Cf(t) + D(t)$, and introduce $C_m = f(t_m) - f_m$, for $m = 0,1,\ldots,M-1$. The discretization of this equation using the method results in the following formulation

$$f_{m+1} = \sum_{j=0}^{n} X_n(j)h^j. \quad (10)$$

By employing the recursive relation given in Equation (10), the generic term $Y_m(j)$ can be written as

$$Y_m(j) = \frac{1}{i!}C^n Y_m(0) + \sum_{p=0}^{j-1}\frac{p!}{j!}F(p). \quad (11)$$

Subtracting Equation (10) from Equation (11), and applying the generic part derived from Equation (11) result in

$$\in_{m+1} = e^{hC} \in_m + h\rho_m. \quad (12)$$

Utilizing the equality

$$\sum_{j=0}^{n}\frac{1}{j!}C^j h^j = e^{hC}. \quad (13)$$

Equation (11) can be expressed as

$$\in_{m+1} = e^{hC} \in_m + \zeta_m \quad (14)$$

In this context, $\zeta_m = h\rho_m$. It's important to observe that the process is deemed compatible with order p if $\zeta_m = O\left(h^{p+1}\right)$. To establish a relation between the global discretization error, the starting error ε_0, and the local discretization error, we can express the recursive relation (14) in the following manner

$$\in_{m+1} = e^{nhC} \in_0 + \sum_{p=0}^{m-1} e^{(m-p-1)hC}\zeta_p. \quad (15)$$

Moreover, the stability of the present method hinges on constraining the term e^{mhC} within the bounds of $mhC \le T$. Therefore, considering the stability criterion expressed as

$$e^{nhC} \le S, \quad (16)$$

for all $m \ge 0,$ and $mh \le T$, the norm bound for the error is given by

$$\|\varepsilon_m\| \le S\|\varepsilon_0\| + S\sum_{i=0}^{m-1}\|\xi_i\|. \quad (17)$$

Following this, employing the definition of the local discretization error, $\|\zeta_i\| \le Rh^{k+1}, \forall i$, and assuming $t(0) = t_0$, we deduce the subsequent inequality for error norm:

$$\|f(t_m) - f_m\| \le R^* h^i. \quad (18)$$

Here, $R = SRt_m$ for all $m = 0,1,\ldots,M$.

4. LDTM for the Nonlinear Duffing Oscillator

We examine the cubic nonlinear Duffing equation, taking into consideration the influence of damping:

$$y'' + \alpha y + \beta y + \gamma y^3 = 0,\ \forall\ 0 \le t \le T, \quad (19)$$

where the starting position, and $v(t)$ are denoted as: $y(0) = y_0$ and $v(0) = y_0$. The constants α, β, and γ are predetermined coefficients. By introducing new functions $y_1(t) = y(t)$, and $y_2(t) = v(t)$, Equation (19) can be recast as the subsequent set of differential equations

$$y_1'(t) = y_2(t), \quad (20)$$

$$y_2'(t) = -\alpha y_2(t) - \beta y_1(t) - \gamma y_1^3(t). \quad (21)$$

Upon applying the DT to Equations (20) and (21), we obtain the following form

$$Y(j+1) = \frac{1}{j+1}\left(CY(j) + D_j\right), \quad (22)$$

where the vector $Y(j) = \left[Y_1(\mathrm{j}),\ Y_2(\mathrm{j})\right]^T$ represents the DT of $\left[y_1(\mathrm{t}),\ y_2(\mathrm{t})\right]^T$, and

$$B_j = \left[0, -\gamma \sum_{l=0}^{j} \sum_{m=0}^{l} Y_2(m) Y_2(l-m) Y_2(j-l) \right]^T.$$

The matrix C can be written as

$$C = (0 \quad 1\text{-}\beta \quad -\alpha). \tag{23}$$

The time interval $[0, T)$ is divided into M sub domains with evenly distributed grid points $0 = t_0 < t_1 < \ldots < T_f$,

where $t_{j+1} = t_j + dt$, and $dt = \frac{T_f}{M}$. In the initial subinterval, the function $y(t)$ can be estimated as $y_0(t)$, defined as follows

$$y_0(t) = Y_0(0) + Y_0(1)t + Y_0(2)t^2 + \ldots + Y_0(j)t^j. \tag{24}$$

With the recurrence relation:

$$Y_0(j+1) = \frac{1}{j+1}\left(CY_0(j) + D_j\right), \tag{25}$$

here, $X_0(0) = [y_0, y_0)$ denotes the initial approximation, and N is the order of the differential transformation. The computed value of the target function $y(t)$ at $t = t_1$ can be expressed as

$$y(t_1) = y_0(t_1) = \sum_{n=0}^{N} Y_0(n) dt^n. \tag{26}$$

The fundamental principle underlying the LDTM is that the obtained solution serves as the initial value for the subsequent iteration. Specifically, $y_1(t_1) = U_1(0) = y_0(t_1)$. Consequently, the approximate solution $y_1(t)$ for the 2^{nd} subspace is formulated as

$$y_1(t) = Y_1(0) + Y_1(1)(t-t_1) + Y_1(2)(t-t_1)^2 + \ldots + Y_1(N)(t-t_1)^j. \tag{27}$$

The coefficients $Y_1(j)$ of the function $y_1(t)$ can be obtained using the same recursive relation as in Equation (25), with the initial condition $Y_1(0) = x_0(t_1)$. Consequently, the approximated solution $y(t_2)$ can be articulated as

$$y(t_2) = y_1(t_2) = \sum_{j=0}^{n} Y_1(j) dt^j. \tag{28}$$

Therefore, the approximate solution at the grid point t_{i+1} can be expressed as

$$y(t_{j+1}) = y_j(t_{j+1}) = \sum_{j=0}^{n} Y_j(j) dt^j. \tag{29}$$

Here, the subscripts range is $j = 0, 1, 2, \ldots, M-1$.

5. Modify LDTM Method

Considering the system of ODE for outlining the implementation procedure of the LDTM:

$$\frac{dy(t)}{dt} = Cy(t) + D(t) \text{ for } 0 \le t \le T, \tag{30}$$

and initial condition $y(0) = y_0$, herein, $y = (y_1, y_2, \ldots, y_n)$, where C and $D(t)$ represent the column vectors of dimensions $n \times 1$. Additionally, C denotes an $m \times m$ matrix. The interval $[0, T)$ is divided into M sub domains, each marked by equidistant grid points, specifically defined as $0 = t_0 < t_1 < \ldots < t_M = T.$ This partitioning guarantees that consecutive points are separated by a constant interval $h,$ where $h = \frac{T}{M}$.

$$Y_n(j+1) = \frac{1}{j+1}\left(CY_m(j) + F(j)\right), \tag{31}$$

where $Y(j)$, and $F(j)$ are the transformed functions of $y(t)$, and $D(t)$, respectively. For the initial sub domain, the function $y(t)$ can be approximated by $y_0(t)$ as follows

$$y_0(t) = Y_0(0) + Y_0(1)t + Y_0(2)t^2 + \ldots + Y_0(j)\frac{t^j}{j}, \tag{32}$$

where j is the order of differential transformation, $Y_0(0) = y_0$, and $Y_0(j)$ can be determined from iteration Equation (31). The approximate value of the dependent variable $y(t)$ at $t = t_1$ can be expressed as follows

$$y(t_1) \approx y_0(t_1) = y_1 = \sum_{j=0}^{K} Y_0(j) h^j. \tag{33}$$

The basic idea in constructing the LDTM is that the obtained approximate solution serves as the initial value for the subsequent iteration, i.e., $y_1(t_1) = Y_1(0) = x_0(t_1)$. Consequently, the approximate solution $y_1(t)$ for the 2^{nd} subspace can be written as

$$y_1(t)=Y_1(0)+Y_1(1)(t-t_1)+Y_1(2)(t-t_1)^2+\ldots+Y_1(j)(t-t_1)^j, \tag{34}$$

and the approximate solution $x_1(t)$ will then be

$$y(t_2)\approx y_1(t_2)=y_2=\sum_{j=0}^{K}Y_1(j)h^j. \tag{35}$$

Therefore, we have at grid point t_{j+1}

$$y(t_{n+1})\approx y_m(t_{m+1})=y_{m+1}=\sum_{j=0}^{K}Y_m(j)h^j+\rho_1, \tag{36}$$

where $\rho_1=\frac{h}{2!}\frac{d}{dt}F(y,)t_{t=t_c}, t_c\in(t_m,t_{m+1}),$ and the subscript is taken to be $m=0,1,2,\ldots,M-1$. Now by using the Rolle's theorem, we have $F(y,t)=\frac{F(y,t_{m+1})-F(y,t_m)}{t_{m+1}-t_m}.$

6. Error Analysis

Consider the subsequent first-order ODE is

$$y(t)=F(y(t),t), t>0, y(0)=y_0, \tag{37}$$

where $y_0\in R^n$, and $F:R\times R^n\to R^n$. The Equation (37) can be represented as follows

$$Y_m(j+1)=\frac{1}{j+1}\left[F(j,t_m)\right), j=0,1,2,\ldots,M-1, m=0,1,2,\ldots,M-1, \tag{38}$$

where

$$y(t_{m+1})\approx y_m(t_{m+1})=y_{m+1}=\sum_{j=0}^{n}Y_n(j)h^j,$$

and the subscript j is chosen to be $j=0,1,2,\ldots,M-1$, where Y_m, and F represent the transformed vector-valued functions. The exact solution of Equation (37) at $t=t_{m+1}$ can be expressed as follows

$$y(t_{j+1})=\sum_{j=0}^{\infty}Y_m(j)h^j=\sum_{j=0}^{n}Y_m(j)h^j+O(h^{j+1}),$$

where $m=0,1,2,\ldots,M-1$ and ρ_m is the local truncation error. The local truncation error can be expressed by using the residual formula of the Taylor series as follows

$$\rho_m=X_m(j+1)h^j$$
$$=\frac{h}{j}\left[F(\mathrm{k},\mathrm{t}_f)=\frac{h}{j}\frac{1}{(j+1)!}\frac{d^jG(y(t),t)}{dt^j}\right)_{t=t_f}, t_c\in(t_m,t_{m+1}), j=2\ldots M-1. \tag{39}$$

7. Results and discussion

This section is dedicated to the numerical demonstration of the proposed method, presenting quantitative and qualitative results through numerous test problems. The accuracy of the obtained results is assessed using error norms and point wise solutions. Comparisons are made with the literature, exact solutions, the RK4 method, and different versions of the modified LDTM. For evaluating the error norms of the current results, we adopt the following norm definition.

Problem [14]: Considering the second-order ODE is

$$y''(t)-2y'(t)+2y(t)=e^t sin(t), 0\le t\le a \tag{40}$$

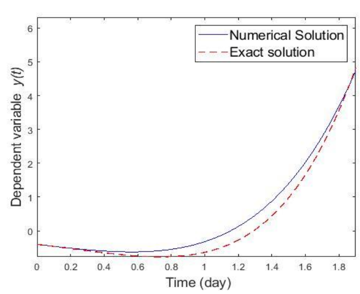
with $y(0)=-0.4, y'(0)=-0.6$. Let us consider $y_1(t)=y(t)$, and $y_1(t)=y_2(t)$. Therefore, Equation (40) can be written as

$$y_2'(t)=-2y_1(t)+2y_2(t)+e^{2t}sin(t); y_1(0)=-0.4, y_2(0)=-0.6. \tag{41}$$

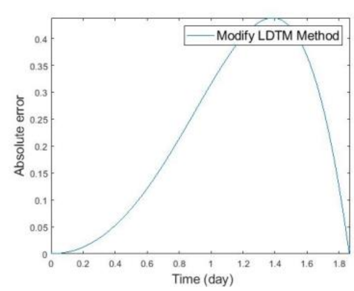
In matrix notation, we have

$$Y'=CY+D, \tag{42}$$

where $Y=(y_1,y_2)$ and $Y(0)=(-0.4,-0.6)$.



(a)



(b)

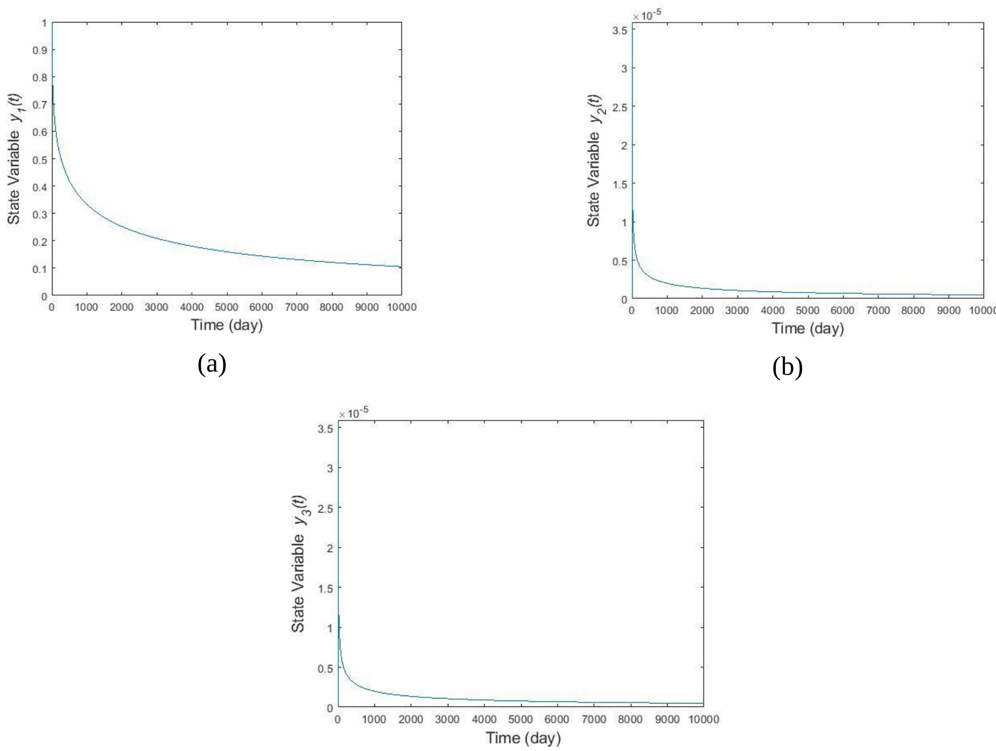
Figure 1: Graphical solution and absolute of the Problem 1.

In above Problem, we have employed a modified LDTM method to assess the absolute error. Within the range of 0 to 1.4, the absolute error exhibits exponential growth. This indicates that as time or the variable progresses from 0 to 1.4, the error amplifies rapidly. Beyond the threshold of $t = 1.4$, the absolute error undergoes a reduction. However, it no longer follows an exponential decline. Instead, it may decelerate or adopt a different pattern. The error reaches its minimum point at x = 1.8. This signifies that within this specific context, the process or function attains its highest level of accuracy, with minimal error occurring at this particular value of *x*.

Problem [15]: Modified Robertson chemical stiff system of nonlinear ODEs is

$$\begin{cases} y'_1(t) = -0.04y_1(t) + 10^4 y_2 y_3 \text{-} 0.96e^{-t}, \\ y'_2(t) = 0.04y_1 \text{-} 10^4 y_2 y_3 \text{-} 3.10^7 y_2^2 \text{-} 0.043^t, \\ y'_3(t) = 3.10^7 y_2^2 + e^{-t}, \end{cases} \tag{43}$$

where $t \in [0, 10000)$ and $y_1(0) = 1,\ y_2(0) = y_3(0) = 0.$ The nonlinear Equation (43) has the exact solution $y_1(t) = e^{-t}, y_2(t) = 0$ and $y_3(t) = 1\text{-}e^{-t}.$



(a) (b) (c)

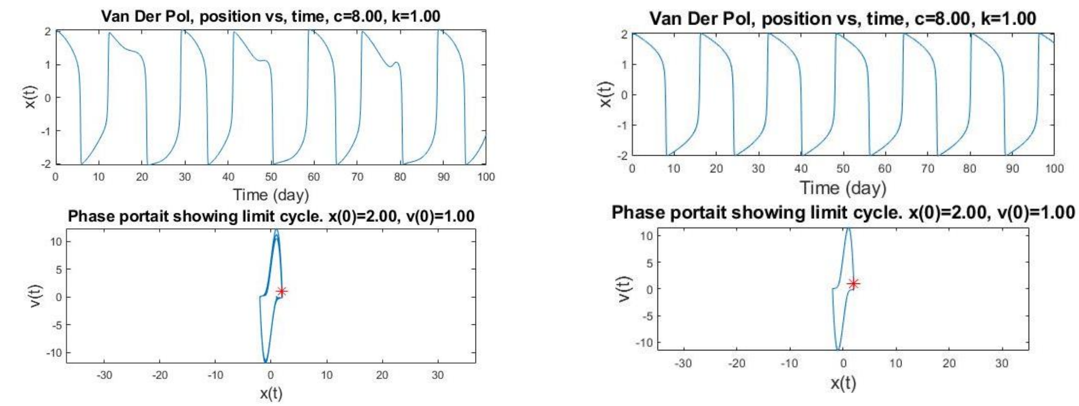
Figure 2: Graph of Robertson chemical equation.

In practical scientific modeling, a multitude of differential equation systems pose a formidable challenge when addressed through traditional numerical methods. This challenge is notably disconcerting because chemical systems like the Robertson model, known for their stiffness, demonstrate substantial instability when approximated using conventional numerical techniques. As anticipated, our examination of the robustness indicator plot affirms the high stiffness of the Robertson Problem (RP). Moreover, our results closely correspond to those derived from the modified LDTM solution. It's worth noting that contemporary ODE-solving algorithms are equipped with automatic step size adjustment and the capability to compute solutions at intermediary points through interpolation.

Problem [16]: In dynamics, the Van Der Pol oscillator is a non-conservative system featuring nonlinear damping. Its time evolution is governed by a second ODE

$$x''(t) - c\left(1\text{-}x^2\right)x'(t) + kx - A\sin(\omega t) = 0. \qquad (44)$$

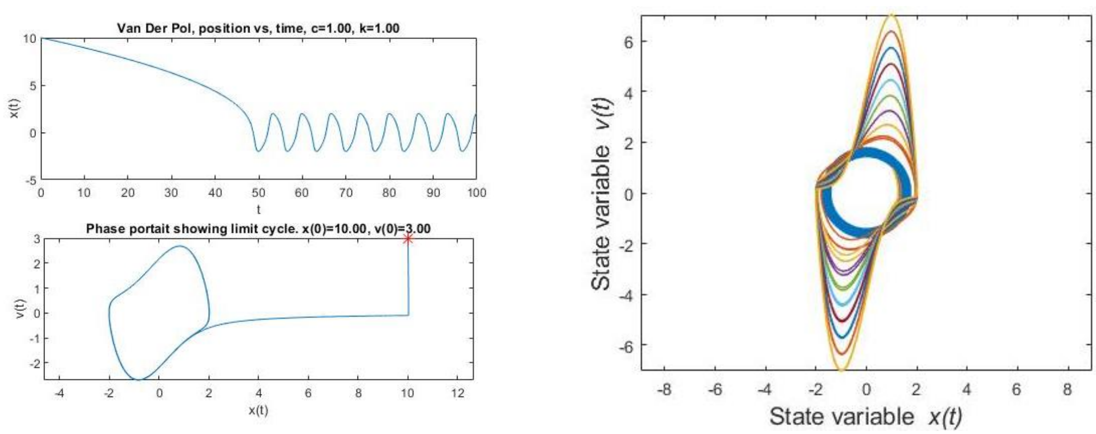
Chaotic behavior in the Van Der Pol oscillator with sinusoidal forcing. The nonlinear damping parameter is equal to *c = 1* and *k = 1*, while the forcing has amplitude $A = 1.0,\ A = 1.2,$ and $\omega = 2\pi/10.$



(a) (b)

Figure 3: Chaotic behavior (a) *c = 1, k = 1,* and *A =1 (b) c = 1, k = 1,* and *A = 1.2.*

If $A = 0,$ the equation (44) becomes $x''(t) - c\left(1\text{-}x^2\right)x'(t) + kx = 0.$ (45)



(a) (b)

Figure 4: Chaotic behavior for the parameter *c = 1,* and *k = 1*.

Problem [9]: The Duffing equation (or Duffing oscillator) is

$$x''(t)+\delta x'(t)+\alpha x+\beta x^3-\gamma cos(\omega t)=0, \tag{46}$$

which is a non-linear second-order differential equation.

α, β, γ : Parameters that characterize the system's characteristics,

δ : Damping, which can be positive or negative,

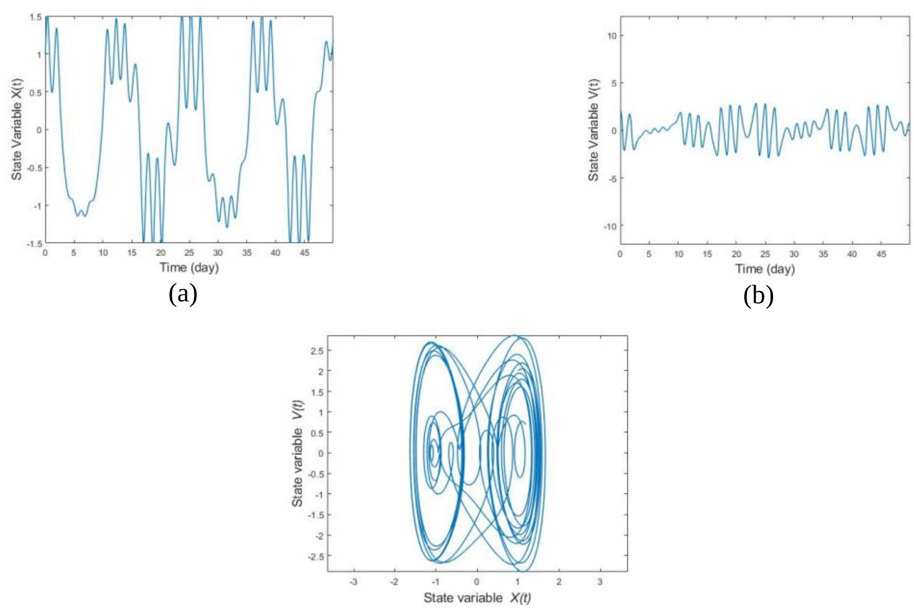
ω : Angular frequency of the external propelling force, $cos(\omega t)$.

In this case, we are interested in a flip bifurcation. A flip bifurcation is a type of bifurcation in which the periodic solution changes from one that oscillates about the origin to one that oscillates about a different equilibrium point. It's also known as a period-doubling bifurcation.

Problem [17]: We have already published a COVID-19 mathematical model. The COVID-19 model is

$$\begin{cases} \dfrac{dS(t)}{dt} = N-\{\mu S+\beta S(E+I+qA+Q+H)\}, \\ \dfrac{dE(t)}{dt} = \beta S(I+qA+Q+H)-kE, \\ \dfrac{dA(t)}{dt} = k(1\text{-}k)\rho E-\gamma_1 A, \\ \dfrac{dI(t)}{dt} = k\rho E-(\gamma_2+\sigma)I, \\ \dfrac{dQ(t)}{dt} = \sigma I-(\alpha+\gamma_3)Q, \\ \dfrac{dH(t)}{dt} = \alpha Q-(\gamma_4+\delta)H, \\ \dfrac{dR(t)}{dt} = \gamma_1 A+\gamma_2 I+\gamma_3 Q+\gamma_4 H, \\ \dfrac{dD(t)}{dt} = \delta H. \end{cases}$$

The constant $\rho(0<\rho<1)$ represents the ratio of exposed $E(t)$ at growth rate k to the clinically infectious sector $I(t)$, while the remaining $(1\text{-}\rho)k$ contributes from $E(t)$ to the sector $A(t)$. The rates of recovery from the $A(t)$, I(t), Q(t), and $H(t)$ sectors are denoted as γ_1, γ_2, γ_3, and γ_4, respectively. The constants α and δ represent the rate of hospitalization originating from $Q(t)$ and the death rate originating from $H(t)$.



(a) (b) (c)

Figure 5: Chaotic behavior in the Duffing equation for γ = 8, ω = 0.5, δ = 0.2, α = 1, β = 5.

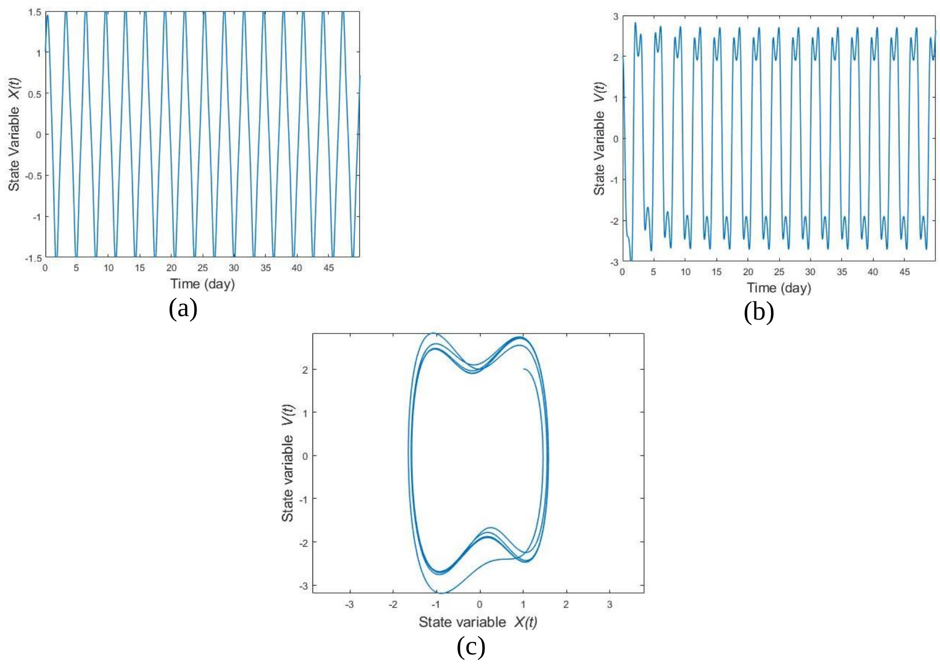


Figure 6: Chaotic behavior in the Duffing equation for γ = 8, ω = 2, δ = 1, α = 1, β = 5.

If $\gamma = 0$, the equation (44) becomes $x''(t) + \delta x'(t) + \alpha x + \beta x^3 = 0.$ (47)

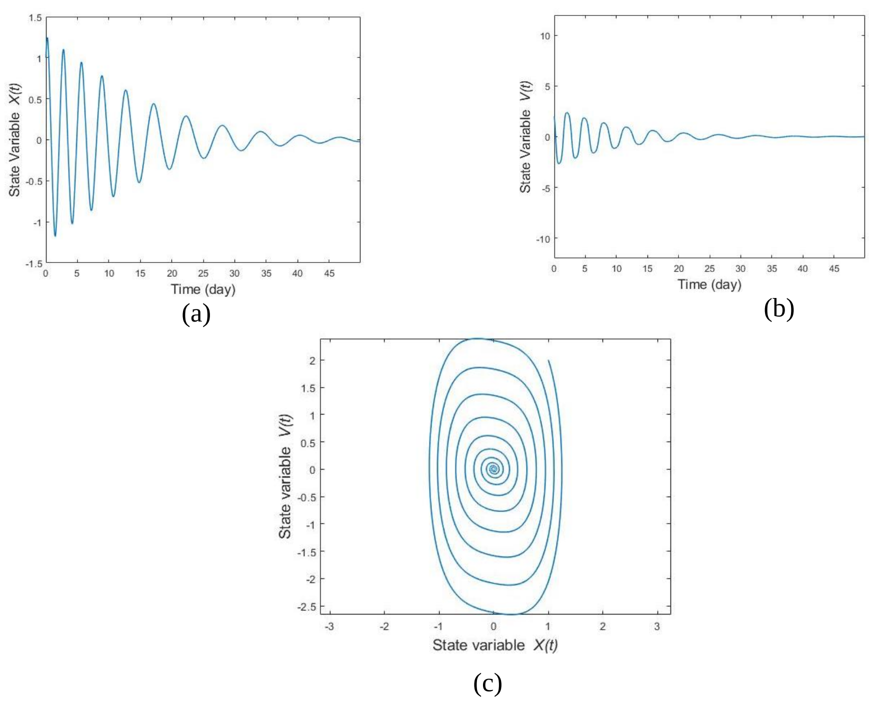
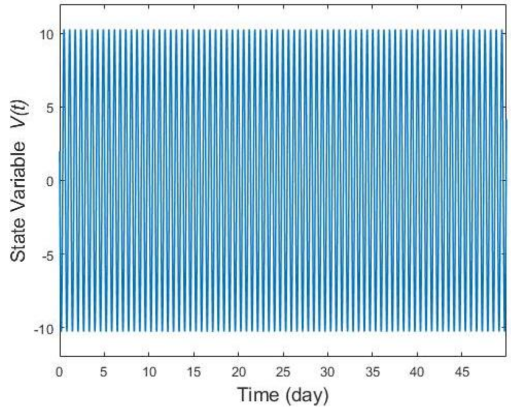
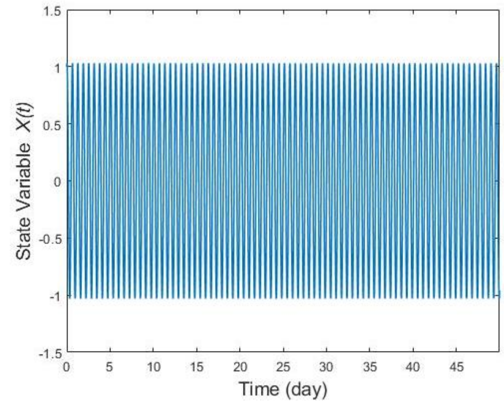


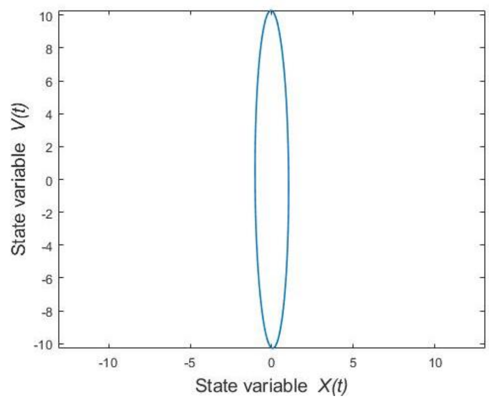
Figure 7: The chaos scenarios of state variables and time, and a phase portrait.



(a)



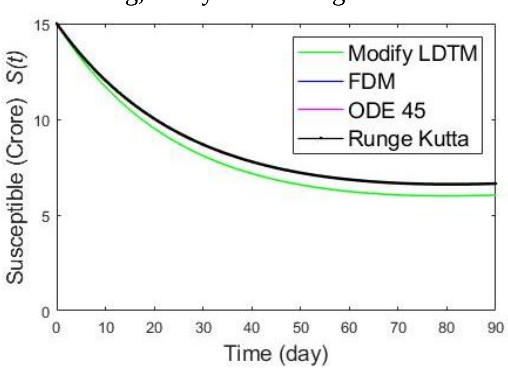
(b)



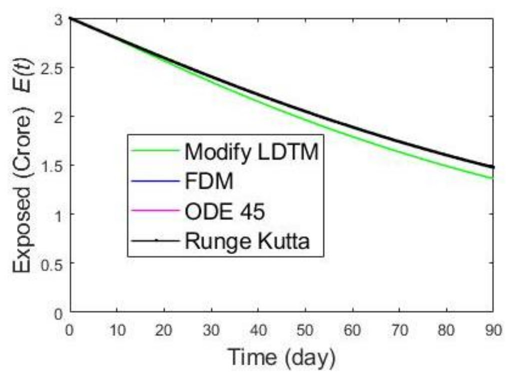
(c)

Figure 8: The flip bifurcation, and phase portrait.

As we gradually increase the amplitude γ of the external forcing, the system undergoes a bifurcation.



(a)



(b)

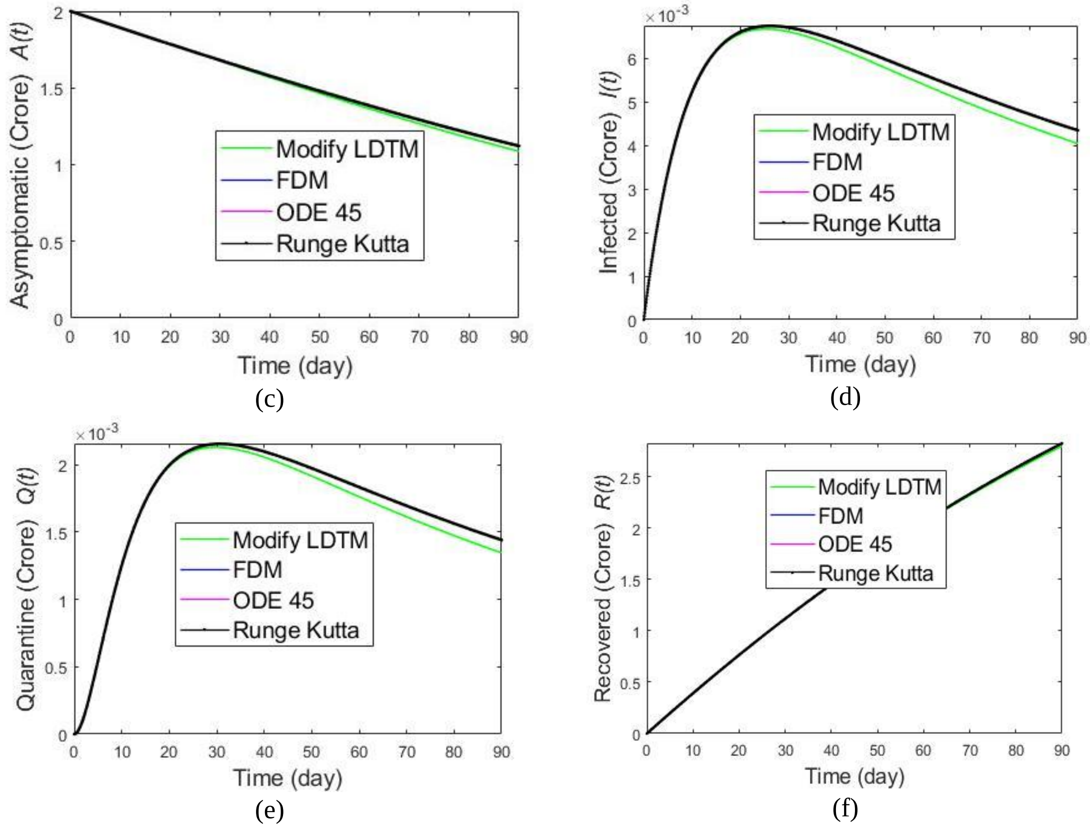


Figure 9: Scenario of state variables of COVID-19.

Problem [14]: Consider the third-order ODE

$$y'''(t)+2y''(t)-y'(t)\text{-}2\text{y}(t)=e^{t},0\le t\le a;y(0)=1,v(0)=2,v'(0)=0. \tag{49}$$

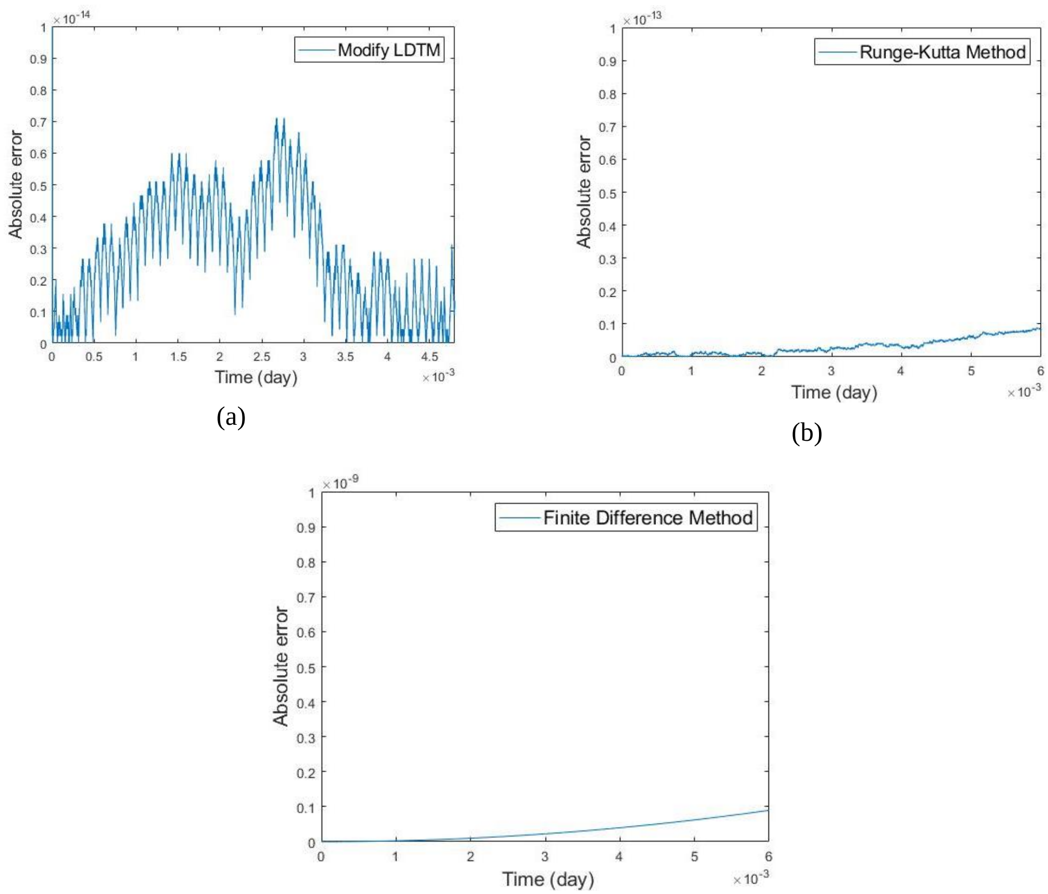
Solution: Put $y_1(t)=y(t),y_2(t)=v(t),$ and $y_3(t)=v'(t)$, equation (49) can be written as

$$\text{y}'_1(t)=y_2(t), \tag{50}$$

$$\text{y}'_2(t)=y_3(t), \tag{51}$$

$$\text{y}'_3(t)=2\text{y}_1(t)+y_2(t)\text{-}2\text{y}_3(t)+e^{t}, \tag{52}$$

with the initial conditions $y_1(0)=1,y_2(0)=2,$ and $y_3(0)=0.$



(a) (b) (c)

Figure 10: Error scenario of Modify LDTM, Runge-Kutta Method, Finite Difference Method.

The modified LDTM is designed to have favorable stability and accuracy properties for certain types of differential equations. In some cases, it can minimize errors over time, especially when dealing with stiff or oscillatory problems. Modified LDTM often utilize adaptive time-stepping schemes, which can dynamically adapt the step size to maintain accuracy. The characteristic of numerical methods for solving differential equations can vary significantly depending on the problem's characteristics, the choice of method, and the step size. The modified LDTM is designed to perform well under certain conditions, but the most suitable method is chosen for the specific problem we solve, and numerical stability and step size are considered to minimize errors over time.

8. Conclusion

The nonlinear Duffing equations, which characterize the dynamics of oscillators subjected to external forces with a spring exhibiting returning force, have been effectively addressed using the modified LDTM. Given that exact solutions for this equation are lacking in the literature across various initial conditions, an exact solution was initially obtained for specific parameters. Moreover, the derivation of the exact solution for the Duffing equation can be regarded as a significant achievement. The modified LDTM has demonstrated its capability to generate remarkably accurate outcomes when compared to alternative methodologies. Notably, this innovative approach excels in capturing the nonlinear behavior of the physical methods, even over extended time periods, in contrast to the competing process which primarily excel over shorter periods. It's important to emphasize that the proposed approach offers a compelling alternative for achieving a high level of precision in the analysis of processes modeled by this equation. Looking ahead, forthcoming research could concentrate on refining and adapting this technique for more intricate nonlinear models, particularly those encompassing intricate forcing terms, thereby broadening its applicability to a wider array of physical processes.

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