Jagannath University Journal of Science
Volume 07, Number II, June 2021, pp. 93-96 https://jnu.ac.bd/journal/portal/archives/science.jsp

# Modification of Gauss Quadrature Formula in Non-symmetric form for Evaluating Singular Integrals 

Research Article

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Received: 03 November 2020
Accepted: 26 January 2021


#### Abstract

Gauss quadrature formula has been modified in a non-symmetric form by introducing a small parameter and applied to evaluate singular integrals. The modified formula measures more accurate result than the original form presented by Gauss.


## Keyword: Singular integral • Gauss's quadrature formulae •Romberg scheme

## 1. Introduction

The evaluation of singular integrals which are arising in many physical, engineering and biological problems is a challenging job. Several authors handle these integrals in different ways (Stewart 1960, Piessens 1970, Stolle et al., 1992). The closed form Newton-Cotes formulae are widely used tools for evaluating definite integrals; but not used directly for singular cases (Sastry 1995). Delves evaluated the principal value of singular integrals by Newton-Cotes and modified Gaussian methods Gaussian methods (Delves, 1968) and Hunter used modified Trapezoidal rule (Hunter 1973); but all these formulae are not straightforward. Gauss's quadrature formula may an approximate technique for evaluating such singular integrals. To obtain better results depending on the nature of functions and the limits, Gauss quadrature formulae were modified and extended as Gauss-Chebyshev, Gauss-Lobbato, Clenshaw-Curtis, Gauss-Kronrod quadrature formulae etc. (Jain et al., 2014). However all previous extended and modified versions of Gauss's formulae are also symmetric. Barden modified Clenshaw-Curtis quadrature (Barden 2013), Dash and Das used mixed quadrature rule and Gauss-Legendre

[^0]quadrature rules to evaluate various integrals (Dash et al., 2011). Fox used combination technique of some elementary processes and Romberg's method to evaluate singular integrals (Fox 1967). Recently, Huq et al. (2011), Hasan et al. (2013), (2014), (2015), (2017), (2018) and Rahaman et. al. (2015), (2020) derived unequal spaced different order straightforward formulae for evaluating singular integrals. In this article, Gauss's nodes have modified shifting with small quantities. The modified formula is non-symmetric and provides better result than Gauss original formula.

## 2. Methodology

### 2.1 Derivation of Gauss Quadrature third order formula:

In general the interval of the Gauss quadrature formula is $[-1,1]$. In this article we have derived Gauss quadrature third order formula in the interval $[0,1]$ using Lagrange's fourth order interpolation formula. By considering five unequal points $x_{0}, x_{1}, \cdots, x_{4}$ together with $x_{0}=0, x_{1}=$ $x_{0}+a, x_{2}=x_{0}+\frac{1}{2}, x_{3}=x_{0}+b$, and $x_{4}=x_{0}+1$ and the Lagrange's formula

$$
\begin{align*}
& y(x)= \frac{\left(x-x_{1}\right)\left(x-x_{2}\right)\left(x-x_{3}\right)\left(x-x_{4}\right) y_{0}}{\left(x_{0}-x_{1}\right)\left(x_{0}-x_{2}\right)\left(x_{0}-x_{3}\right)\left(x_{0}-x_{4}\right)}+\frac{\left(x-x_{0}\right)\left(x-x_{2}\right)\left(x-x_{3}\right)\left(x-x_{4}\right) y_{1}}{\left(x_{1}-x_{0}\right)\left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{1}-x_{4}\right)} \\
&+\frac{\left(x-x_{0}\right)\left(x-x_{1}\right)\left(x-x_{3}\right)\left(x-x_{4}\right) y_{2}}{\left(x_{2}-x_{0}\right)\left(x_{2}-x_{1}\right)\left(x_{2}-x_{3}\right)\left(x_{2}-x_{4}\right)}+\frac{\left(x-x_{0}\right)\left(x-x_{1}\right)\left(x-x_{2}\right)\left(x-x_{4}\right) y_{3}}{\left(x_{3}-x_{0}\right)\left(x_{3}-x_{1}\right)\left(x_{3}-x_{2}\right)\left(x_{3}-x_{4}\right)}+ \\
& \quad \frac{\left(x-x_{0}\right)\left(x-x_{1}\right)\left(x-x_{2}\right)\left(x-x_{3}\right) y_{4}}{\left(x_{4}-x_{0}\right)\left(x_{4}-x_{1}\right)\left(x_{4}-x_{2}\right)\left(x_{4}-x_{3}\right)}, \tag{1}
\end{align*}
$$

where $y_{0}, y_{1}, \cdots, y_{4}$ are functional values at $x_{0}, x_{1}, \cdots$, $\mathrm{X}_{4}$.
Integrating Eq. (1) with respect to x from $\mathrm{x}_{0}=0$ to $\mathrm{x}_{4}=$ 1 and equating the coefficient of $y_{0}$ and $y_{4}$ to zero we obtain $\mathrm{a}=\frac{1}{10}(5-\sqrt{15})$ and $\mathrm{b}=\frac{1}{10}(5+\sqrt{15})$. The curve joining the above points according to Lagrange's fourth order interpolation formula (Eq. (1)) and integrating with respect to x from $\mathrm{x}_{0}$ to $\mathrm{x}_{4}$ we obtain the formula as
$I=\int_{x_{0}}^{x_{4}} y(x) d x=\int_{0}^{1} y(x) d x=\frac{1}{18}\left(5 y_{1}+8 y_{2}+5 y_{3}\right)$,
It is noted that the weights of the Gauss's third order formula are $\frac{5}{9}, \frac{8}{9}$ and $\frac{5}{9}$ and the interval is $[-1,1]$. The weights of Eq. (2) become half of Gauss's original formula since the interval has considered $[0,1]$. However, the formula can be easily extended to in an interval $[0, \mathrm{~h}], \mathrm{h}>0$ using the same interpolation formula by considering five points $x_{0}, x_{1}=x_{0}+h a$, $x_{2}=x_{0}+\frac{h}{2}, x_{3}=x_{0}+h b$, and $x_{4}=x_{0}+h$. The Gauss's quadrature third order formula becomes

$$
\begin{equation*}
\mathrm{I}=\int_{\mathrm{x}_{0}}^{\mathrm{x}_{4}} \mathrm{y}(\mathrm{x}) \mathrm{dx}=\frac{\mathrm{h}}{18}\left(5 \mathrm{y}_{1}+8 \mathrm{y}_{2}+5 \mathrm{y}_{3}\right) \tag{3}
\end{equation*}
$$

whether $x_{0}$ becomes zero or not. The above formula is very important for utilization of Romberg integration.

### 2.2 Derivation of the modified third order formula:

In order to derive the modified formula, we consider five unequal points $\mathrm{x}_{0}=0, \mathrm{x}_{1}=\mathrm{a}-\mathrm{p} \varepsilon, \mathrm{x}_{2}=\frac{1}{2}-\varepsilon, \mathrm{x}_{3}=$ $\mathrm{b}+\mathrm{q} \varepsilon, \mathrm{x}_{4}=1$ where $0<\varepsilon \ll 1$ and the values of aand b are given in previous. Now integrating Eq. (1) with respect to x from $\mathrm{x}_{0}=0$ to $\mathrm{x}_{4}=1$ and equating the coefficient of $y_{0}$ and $y_{4}$ to zero we obtain $p=\frac{2}{5}$ and $q=-\frac{2}{5}$. Thus the modified formula becomes
$I=\int_{x_{0}}^{x_{4}} y(x) d x=\frac{1}{18}\left(5 y_{1}+8 y_{2}+5 y_{3}\right)-\frac{2 \sqrt{15}}{9}\left(y_{1}-\right.$ $\left.\mathrm{y}_{3}\right) \varepsilon$.
If $\varepsilon=0$, formula Eq. (4) reduces to Gauss's formula Eq. (2). The value $\varepsilon$ is to be determined from error terms of this formula. The modified formula also be extended to in an interval [0, h], h > 0as
$I=\int_{x_{0}}^{x_{4}} y(x) d x=\frac{h}{18}\left(5 y_{1}+8 y_{2}+5 y_{3}\right)-$
$\frac{2 \sqrt{15} h}{9}\left(\mathrm{y}_{1}-\mathrm{y}_{3}\right) \varepsilon$.
The formula (3) and (5) excluded $\mathbf{y}_{0}$ and $\mathbf{y}_{4}$, so they are directly applicable when lower, upper or/and both singularities arise.

## 3. Error of Gauss's and the proposed formula:

The error of Gauss's third order formula and modified formula are respectively

$$
\begin{equation*}
E_{G}=\frac{h^{7} F^{(6)}(\xi)}{2016000}-\frac{h^{8} F^{(7)}(\xi)}{4032000} \cdots \tag{6}
\end{equation*}
$$

and
$\mathrm{E}_{\mathrm{M}}=\frac{\mathrm{h}^{7} \mathrm{~F}^{(6)}(\xi)}{2016000}-\frac{\mathrm{h}^{8} \mathrm{~F}^{(7)}(\xi)}{4032000}+\frac{\mathrm{h}^{6} \varepsilon \mathrm{~F}^{(5)}(\xi)}{12000}-\frac{\mathrm{h}^{7} \varepsilon \mathrm{~F}^{(6)}(\xi)}{24000}+$ $\frac{37 \mathrm{~h}^{8} \varepsilon \mathrm{~F}^{(7)}(\xi)}{3360000}+\cdots$
where $x_{0}<\xi<x_{0}+h$. The value of $\varepsilon$ is determined from the above expression considering a minimum value.

## 4. Examples:

4.1 Let us consider a singular integral

$$
\begin{equation*}
\mathrm{I}=\int_{0}^{1} \frac{1}{\sqrt{\mathrm{x}}} \mathrm{dx} \tag{8}
\end{equation*}
$$

Herein $x=0$ is the singular point and exact value of this integral is 2 . The approximate values obtained by Gauss's formula and proposed modified formula have been presented in Table 1. The Romberg schemes obtained by Gauss's third order formula and proposed modified third order formula have been presented in Table 2 and Table 3 respectively.
Table 1. The comparisons of the results obtained by Gauss's and proposed formula.

| Formulae | Results | Error | Exact |
| :---: | :---: | :---: | :---: |
| Gauss- <br> Legendre | 1.750863177975 | 0.249136822025 |  |
| Proposed | 1.759439975564 | 0.240560024436 | 2 |

Table 2. Romberg scheme obtain by Gauss-Legendre 3rd order formula.

| $G(h)=1.750863177975$ | $G\left(h, \frac{h}{2}\right)$ |
| :--- | :--- |
| $G\left(\frac{h}{2}\right)=1.823828812053$ |  |

Table 3. Romberg scheme obtain by proposed 3rd order formula.

| $N(h)$ <br> $=1.759439975564$ | $N\left(h, \frac{h}{2}\right)$ |
| :--- | :--- |
|  |  |

Example 4.2 Consider a more complicated singular integral with exact value 3 as

$$
\begin{equation*}
I=\int_{0}^{1} x^{-\frac{2}{3}} d x \tag{9}
\end{equation*}
$$

The approximate values of (9) obtained by Gauss's and proposed modified formulae have been presented in
Table 4. The Romberg schemes obtained by GaussLegendre third order formula and proposed modified third order formula have been presented in Table 5 and Table 6 respectively.

Table 4. The comparisons of the results obtained by Gauss's and proposed formula.

| Formulae | Results | Error | Exact |
| :---: | :---: | :---: | :---: |
| Gauss- <br> Legendre | 2.196879076641 | 0.803120923359 |  |
| Proposed | 2.240862463744 | 0.7591375362558703 |  |

Table 5. Romberg scheme obtain by Gauss-Legendre 3rd order formula.

| $G(h)=2.1968790766412094$ |  |
| :---: | :---: |
| $\begin{aligned} & G\left(\frac{h}{2}\right) \\ & =2.3625534210077306 \end{aligned}$ | $\begin{aligned} & G\left(h, \frac{h}{2}\right) \\ & =2.999955987800722 \end{aligned}$ |

Table 6. Romberg scheme obtained by proposed 3rd order formula.

| $\begin{array}{l}N(h) \\ =2.2408624637441297\end{array}$ | $N\left(h, \frac{h}{2}\right)$ |
| :--- | :--- |
| $N\left(\frac{h}{2}\right)$ |  |
| $=2.397471736979085$ |  |$)=2.999998055252643$

## 5. Results and Discussion

A modified Gauss-quadrature formula has been presented to evaluate singular integrals. To illustrate the formula, the approximate solutions of some singular integrals have been compared with corresponding gauss-quadrature solutions.

The results and the error obtained by Gauss-quadrature formula Eq. (3) and proposed formula Eq. (5) are tabulated in Table1 and Table 4. These tables show that the error obtained by the modified formula is less than Gauss-quadrature formula. For Romberg scheme, by Table 2, Table 3 and Table 5, Table 6 it is clear that the proposed modified formula converges faster than original Gauss's quadrature formula.

## 6. Conclusion

From the above observation it may conclude that, the idea of the simple implementation of Gauss's quadrature using Lagrange's interpolation formula, the application of Romberg technique in Gauss's quadrature and finally the modification of Gauss's quadrature in non-symmetric form may attractive to the numerical researchers and engineers in the case of singular integrals.

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