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Numerical Method for 1D Quasi-Linear Hyperbolic Equation on a Graded Mesh in Time: Application to Telegraphic Equation

Research Article

Bishnu Pada Ghosh, Goutam Kumar Saha, Rabindra Nath Mondal*

Department of Mathematics, Jagannath University, Dhaka-1100, Bangladesh.

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Abstract: We develop a novel three level implicit method of order two on a graded mesh in time for the approximation of 1D second order quasi-linear hyperbolic partial differential equation $\phi_{tt} = \alpha(x,t,\phi)\phi_{xx} + R(x,t,\phi,\phi_x,\phi_t)$, $0 < x < 1$, $t > 0$ subject to appropriate initial and Dirichlet boundary conditions. When the method is applied to the Telegraph equation, the method is shown to be unconditionally stable on a graded mesh. We do not need any iterative method to solve the linear difference equations. An explicit method of order two at the first time level is discussed in detail. We have solved five benchmark problems to test the viability of the proposed method. For linear differential equations, we use the Gauss-elimination procedure, whereas for a nonlinear or quasi-linear differential equation, we use the Newton-Raphson method at each advanced time level. The suggested scheme is scrutinized on several physical problems to exhibit the accuracy and effectiveness of the proposed method.

Keywords : *Quasi-linear • Graded mesh • Telegraphic equation • Wave equation • Dissipative*
AMS Classification (2010): *65M06; 65M12; 65M22; 65Y99*

1. Introduction

Let us consider the 1D quasi-linear hyperbolic partial differential equation (HPDE)

$$\frac{\partial^2\phi}{\partial t^2} = \alpha(x,t,\phi)\frac{\partial^2\phi}{\partial x^2} + R(x,t,\phi,\phi_x,\phi_t),$$
$$0 < x < 1, t > 0 \quad (1)$$

with two initial conditions prescribed by

$$\phi(x,0) = f(x), \quad \phi_t(x,0) = g(x),$$
$$0 \le x \le 1, \quad (2)$$

and two boundary conditions are given by

$$\phi(0,t) = \phi_0(t), \quad \phi(1,t) = \phi_1(t), \quad t > 0. \quad (3)$$

We assume that $\alpha(x,t,\phi) > 0$ and α, ϕ are satisfactorily regular, and also their required higher order partial derivatives are defined analytically in the solution region $\Omega \equiv \{(x,t): 0 < x < 1, t > 0\}$.The initial and boundary conditions (2)-(3) are given with required regularity to remain the order of the method unchanged. Besides we presume that there exists exactly one regular solution for the initial boundary value problem (IBVP) (1)-(3). Required information is discussed by Li *et al.* (2007).

The numerical approximation of 1D quasi-linear hyperbolic partial differential equations (HPDEs) plays a significant role in many areas of engineering, mathematical and physical sciences. The displacement of any point of the vibrating string at the position *x* at any time *t* is represented by the function ϕ, which is a function of *x* and *t*. Electromagnetic waves, chemical waves, seismic waves, shallow water and tsunami waves etc. are the examples of well-known physical waves. Linear and non-linear phenomena are described by linear and non-linear hyperbolic equations respectively. Hence, we cannot apply the superposition principle to the non-

*Corresponding author: Rabindra Nath Mondal
E-mail: *rnmondal71@yahoo.com*

linear wave equations. It has been experienced in the past that the non-linear hyperbolic partial differential equations are more complicated to solve analytically and there are no general methods exist for the solution of such equations. Therefore, stable numerical methods are the only choice to handle such problems. Greenspan (1968) introduced the boundary value technique to obtain an approximate solution of the wave equation. Ciment and Leventhal (1975, 1978) studied fourth order compact implicit method for solving wave equation. Any explicit scheme for second order hyperbolic equation is stable for a certain stability range. However, Twizell (1979) has introduced a new compact scheme for the wave equation with an improved stability interval. Using additive operator technique, Mohanty (2007) has established stability range for explicit techniques for multi-dimensional HPDEs with first order space derivative terms.

To obtain stability interval of a numerical method associated with second order hyperbolic type initial boundary value problems is a challenging task for engineers and research scientists. 1D Telegraphic equation narrates a family of the array of many physical systems; e.g., the propagation of current signals and voltage in co-axial transmission lines, the propagation of acoustic waves in Darcy-type porous media and parallel viscous Maxwell fluid flow, etc. Using the techniques for purely initial value problems, many researchers (Mohanty 2004, 2005, Mohebbi *et al.* 2008, Pandit *et al.* 2015, Gao *et al.* 2007, El-Azab *et al.* 2007, Dehghan *et al.* 2008 and Ding *et al.* 2009) have proposed unconditionally stable implicit schemes for the approximation of the Telegraphic equation using uniform grid. To the author's knowledge, no numerical schemes and the corresponding stability analysis for the Telegraphic equation on a variable mesh in time direction have been considered in the literature so far. Using three grid points, numerical scheme for non-linear two-point BVPs on a non-uniform grid have been discussed. Recently, Mohanty *et al.* (2015, 2021a, 2021b) have established the stability interval $(0,\infty)$ for two-steps pure initial value problems on a variable mesh. In the present paper, we discuss two new three-level implicit methods of order of accuracy (i) one in time and two in space, and (ii) two in time and two in space on a variable grid in the *t*-direction for the solution of 1D quasilinear initial-boundary value problems (1)-(3). The paper is segregated as follows: In Section 2, the methods are based on a variable grid in the *t*-direction for the solution of second order quasilinear hyperbolic equations of order one in time and two in space, and two in time and space have been formulated. Section 3 provides the derivation of the numerical methods. In section 4, we discuss the application of the proposed methods to Telegraphic equation and stability analysis on the variable grid. Further in section 5, several hyperbolic equations of physical repute have been computed to elucidate and inspect the accuracy of the suggested methods. Finally, section 6 summarizes all the steps.

2. Conceptualization and Derivation of the numerical methods

Now, we consider the 1D nonlinear HPDE of the form

$$\frac{\partial^2\phi}{\partial t^2} = \alpha(x,t)\frac{\partial^2\phi}{\partial x^2} + R(x,t,\phi,\phi_x,\phi_t),$$
$$0 < x < 1,\ t > 0, \tag{4}$$

where $\alpha(x,t) > 0$ and the initial and boundary conditions are given by (1)-(3). The domain $(0,1)\times(0,\infty)$ is covered with a rectangular grid of uniform grid size $h > 0$ in the *x*-direction, where $x_i = ih, i = 0(1)N+1$; variable grid size $\tau_n = t_n - t_{n-1}$ in *t*-direction, where $t_0 < t_1 < t_2 < \cdots < T$. T is a end point in which the approximate value of ϕ requires to be determined. Assume $\rho = (\frac{\tau_{n+1}}{\tau_n})>0$, $n = 1, 2, 3,\ldots$, so that $\tau_{n+1} = \rho\tau_n$. For $\rho = 1$, the grid sizes in *t*-direction are uniform all over the solution domain.

Let $\Phi_i^n = \phi(x_i,t_n)$ be the analytical solution value of $\phi(x,t)$ at the nodal point (x_i,t_n), and let ϕ_i^n approximates Φ_i^n. Let $\alpha_i^n = \alpha(x_i,t_n)$, ${\alpha_x}_i^n = \alpha_x(x_i,t_n)$, ${\alpha_t}_i^n = \alpha_t(x_i,t_n)$, ${\alpha_{xx}}_i^n = \alpha_{xx}(x_i,t_n)$ be the analytical values of α, α_x, α_t, α_{xx} at the nodal point (x_i,t_n), respectively.

For the derivation of the method, we use Numerov type approximation for uniform mesh in the *x*-direction and variable mesh approximation in the *t*-direction. At the mesh point (x_i,t_n), we may write the nonlinear differential equation (4) as

$${\Phi_{tt}}_i^n - \alpha_i^n{\Phi_{xx}}_i^n = R(x_i,t_n,\Phi_i^n,{\Phi_x}_i^n,{\Phi_t}_i^n) \equiv R_i^n(\text{say}). \tag{5}$$

With the aim to derive finite difference scheme for the HPDE (4), consider the following linear combination:

$$\begin{aligned}P_0R_i^n + P_1hR_{x_i}^n + P_2\tau_nR_{x_i}^n &= P_0\left[{\overline{\Phi}_{tt}}_i^n - \frac{(\rho-1)}{3}\tau_n\Phi_{ttt_i}^n + P_2\tau_n\Phi_{ttt_i}^n\right] - hP_1\alpha_i^n{\Phi_{xxx}}_i^n\end{aligned}$$

$$+hP_1{\overline{\Phi}_{xtt}}_i^n - P_2\tau_n\alpha_i^n{\overline{\Phi}_{xxt}}_i^n - \left[P_0\alpha_i^n + hP_1{\alpha_x}_i^n + P_2\tau_n{\alpha_t}_i^n\right]{\overline{\Phi}_{xx}}_i^n + O(\tau_n^2 + h^2) \tag{6}$$

Equating the coefficients of $\Phi_{xxx_i}^n$ and $\Phi_{ttt_i}^n$ to zero, we get

$P_1 = 0,\ P_2 = \dfrac{(\rho-1)}{3} P_0$

Let $P_0 = 1$, so $P_2 = \frac{(\rho-1)}{3}$.

$$L_\phi \equiv \bar{\Phi}_{tt_i}^{\ n} - \left[\alpha_i^n + \frac{(\rho-1)}{3}\tau_n \alpha_{t_i}^n\right]\bar{\Phi}_{xx_i}^{\ n}$$

$\frac{(\rho-1)}{3}\tau_n \alpha_i^n \bar{\Phi}_{xxt_i}^{\ n} = R_i^n + \frac{(\rho-1)}{3\rho(\rho+1)}[R_i^{n+1}$
$-(1-\rho^2)R_i^n - \rho^2 R_i^{n-1}] + O({\tau_n}^2 + h^2).$ (7)

For variable mesh discretization, we require the following approximations:

$$\bar{\Phi}_{t_i}^{\ n} = \frac{1}{\rho(1+\rho)\tau_n}[\Phi_i^{n+1} - (1-\rho^2)\Phi_i^n - \rho^2\Phi_i^{n-1}] = \Phi_{t_i}^{\ n} + O({\tau_n}^2), \quad (8)$$

$$\bar{\Phi}_{t_i}^{\ n+1} = \frac{1}{\rho(1+\rho)\tau_n}[(1+2\rho)\Phi_i^{n+1} - (1+\rho)^2\Phi_i^n + \rho^2\Phi_i^{n-1}] = \Phi_{t_i}^{\ n+1} + O({\tau_n}^2), \quad (9)$$

$$\bar{\Phi}_{t_i}^{\ n-1} = \frac{1}{\rho(1+\rho)\tau_n}[-\Phi_i^{n+1} + (1+\rho)^2\Phi_i^n - \rho(2+\rho)\Phi_i^{n-1}] = \Phi_{t_i}^{\ n-1} + O({\tau_n}^2), \quad (10)$$

$$\bar{\Phi}_{x_i}^{\ n} = \frac{1}{2h}[\Phi_{i+1}^n - \Phi_{i-1}^n] = \Phi_{x_i}^{\ n} + O(h^4), \quad (11)$$

$$\bar{\Phi}_{x_i}^{\ n+1} = \frac{1}{2h}[\Phi_{i+1}^{n+1} - \Phi_{i-1}^{n+1}] = \Phi_{x_i}^{\ n+1} + O(h^2), \quad (12)$$

$$\bar{\Phi}_{x_i}^{\ n-1} = \frac{1}{2h}[\Phi_{i+1}^{n-1} - \Phi_{i-1}^{n-1}] = \Phi_{x_i}^{\ n-1} + O(h^2), \quad (13)$$

$$\bar{\Phi}_{tt_i}^{\ n} = \frac{2}{\rho(1+\rho){\tau_n}^2}[\Phi_i^{n+1} - (1+\rho)\Phi_i^n + \rho\Phi_i^{n-1}] = \Phi_{tt_i}^{\ n} + \frac{(\rho-1)}{3}\tau_n\Phi_{ttt} + O({\tau_n}^2), \quad (14)$$

$$\bar{\Phi}_{xx_i}^{\ n} = \frac{1}{h^2}[\Phi_{i+1}^n - 2\Phi_i^n + \Phi_{i-1}^n] = \Phi_{xx_i}^{\ n} + O(h^4), \quad (15)$$

$$\bar{\Phi}_{xxt_i}^{\ n} = \frac{1}{\rho(1+\rho)\tau_n h^2}[(\Phi_{i+1}^{n+1} - 2\Phi_i^{n+1} + \Phi_{i-1}^{n+1}) - (1-\rho^2)(\Phi_{i+1}^n - 2\Phi_i^n + \Phi_{i-1}^n) - \rho^2(\Phi_{i+1}^{n-1} - 2\Phi_i^{n-1} + \Phi_{i-1}^{n-1})] = \Phi_{xxt_i}^{\ n} + O({\tau_n}^2 + \tau_n h^2), \quad (16)$$

We require the following approximations for $R(x,t,\phi,\phi_x,\phi_t)$.

Let

$$\bar{R}_i^n = R(x_i, t_n, \Phi_i^n, \bar{\Phi}_{x_i}^{\ n}, \bar{\Phi}_{t_i}^{\ n}) = R_i^n + O(h^2), \quad (17)$$

$$\bar{R}_i^{n+1} = R\left(x_i, t_{n+1}, \Phi_i^{n+1}, \bar{\Phi}_{x_i}^{\ n+1}, \bar{\Phi}_{t_i}^{\ n+1}\right) = R_i^{n+1} + O({\tau_n}^2 + h^2), \quad (18)$$

$$\bar{R}_i^{n-1} = R\left(x_i, t_{n-1}, \Phi_i^{n-1}, \bar{\Phi}_{x_i}^{\ n-1}, \bar{\Phi}_{t_i}^{\ n-1}\right) = R_i^{n-1} + O({\tau_n}^2 + h^2) \quad (19)$$

Then the approximation for the differential equation (4) at each grid point (x_i, t_n) is given by

$$L_\phi \equiv \bar{\Phi}_{tt_i}^{\ n} - [\alpha_i^n + \frac{(\rho-1)}{3}\tau_n\alpha_{t_i}^n]\bar{\Phi}_{xx_i}^{\ n} - \frac{(\rho-1)}{3}\tau_n\alpha_i^n\bar{\Phi}_{xxt_i}^{\ n} = \bar{R}_i^n + \frac{(\rho-1)}{3\rho(\rho+1)}[(\bar{R}_i^{n+1} - (1-\rho^2)\bar{R}_i^n - \rho^2\bar{R}_i^{n-1})] + \bar{T}_i^n\ ;$$

$i = 1(1)\ N$, $n = 0, 1, 2, \ldots$ (20)

where the local truncation error (LTE)

$\bar{T}_i^n = O({\tau_n}^2 + h^2).$

Now using the approximations (8)-(16) and (17)-(19) from (7) and (20), the LTE is obtained as

$$\bar{T}_i^n = O({\tau_n}^2 + h^2). \quad (21)$$

For quasi-linear HPDE (1), that is, whenever $\alpha = \alpha(x,t,\phi)$, we modify the equation (17) using the approximations given below:

$$\alpha_{t_i}^{\ n} = \frac{1}{\rho(1+\rho)\tau_n}[\alpha_i^{n+1} - (1-\rho^2)\alpha_i^n - \rho^2\alpha_i^{n-1}] + O({\tau_n}^2), \quad (22)$$

$$\alpha_{x_i}^{\ n} = \frac{1}{2h}[\alpha_{i+1}^n - \alpha_{i-1}^n] + O(h^2), \quad (23)$$

$$\alpha_{xx_i}^{\ n} = \frac{1}{h^2}[\alpha_{i+1}^n - 2\alpha_i^n + \alpha_{i-1}^n] + O(h^2), \quad (24)$$

where

$\alpha_i^n = \alpha(x_i, t_n, \Phi_i^n),\ \alpha_{i\pm1}^n = \alpha(x_{i\pm1}, t_n, \Phi_{i\pm1}^n),$
$\alpha_i^{n\pm1} = \alpha(x_i, t_{n\pm1}, \Phi_i^{n\pm1}).$

Substituting (22)-(24) into (20), we get the numerical scheme of $O({\tau_n}^2 + h^2)$ for the quasi-linear HPDE (1) and the order of the LTE remains the same. For $\rho = 1$, $\Rightarrow \tau_n = \tau_{n+1} = \tau$, the proposed method (20) becomes of $O(\tau^2 + h^2)$.

Incorporating the prescribed initial and boundary conditions (2)-(3), the method (20) can be expressed in a tri-diagonal matrix form at each advanced time level. For linear differential equations, we use the Gauss-elimination procedure; whereas, for a nonlinear or quasi-linear differential equation, we use the Newton-Raphson method (Kelly 1995 and Hageman *et al.* 2004).

3. Stability consideration

The mathematical modelling of well-known Telegraph equation in 1D with a forcing function is given by

$$\phi_{tt} + 2\alpha_0\phi_t + {\beta_0}^2\phi = \phi_{xx} + f(x,t),\ \alpha_0 > 0, \beta_0 \geq 0, \quad (25)$$

defined in the domain $[0 < x < 1] \times [t > 0]$, where α_0, β_0 are constants. Equation (25) represents a damped wave equation when $\beta_0 = 0$. In this section, we denote $a_n = {\alpha_0}^2{\tau_n}^2, b_n = {\beta_0}^2{\tau_n}^2$ and $p_n = \frac{\tau_n}{h} > 0$.

Applying the scheme (20) to the PDE (25), we get

$\bar{\Phi}_{tt_i}^{\ n} - \bar{\Phi}_{xx_i}^{\ n} - \frac{(\rho-1)}{3}\tau_n \bar{\Phi}_{xxt_i}^{\ n} + \frac{2\alpha_0(\rho-1)}{3\rho(\rho+1)}\Big(\bar{\Phi}_{t_i}^{\ n+1} - (1-\rho^2)\bar{\Phi}_{t_i}^{\ n} - \rho^2\bar{\Phi}_{t_i}^{\ n-1}\Big) + 2\alpha_0\bar{\Phi}_{t_i}^{\ n} + {\beta_0}^2\Phi_i^n$
$+ \frac{{\beta_0}^2(\rho-1)}{3\rho(\rho+1)}\big(\Phi_i^{n+1} - (1-\rho^2)\Phi_i^n - \rho^2\Phi_i^{n-1}\big)\Big] = \sum f + O({\tau_n}^2 + h^2),$ (26)

where

$f_i^n = f(x_i, t_n)$ and $\sum f$
$= f_i^n + \frac{(\rho-1)}{3\rho(\rho+1)}\big(f_i^{n+1} - (1-\rho^2)f_i^n - \rho^2 f_i^{n-1}\big).$

To simplify (26), we use the following:

$\bar{\Phi}_{t_i}^{\ n+1} - (1-\rho^2)\bar{\Phi}_{t_i}^{\ n} - \rho^2\bar{\Phi}_{t_i}^{\ n-1} = \frac{2}{\tau_n}[\Phi_i^{n+1} - (1+\rho)\Phi_i^n + \rho\Phi_i^{n-1}].$ (27)

Multiplying $\frac{\rho(\rho+1)}{2}{\tau_n}^2$ throughout (26), using the relation (27) and simplifying, we get

$\left[1 + \frac{2(\rho-1)}{3}\sqrt{a_n}\right][\Phi_i^{n+1} - (1+\rho)\Phi_i^n + \rho\Phi_i^{n-1}]$
$+ \left[\sqrt{a_n} + \frac{(\rho-1)}{6}b_n - \frac{(\rho-1)}{6}{p_n}^2{\delta_x}^2\right],$
$[\Phi_i^{n+1} - (1-\rho^2)\Phi_i^n - \rho^2\Phi_i^{n-1}]$
$-\frac{\rho(\rho+1)}{2}{p_n}^2{\delta_x}^2\Phi_i^n + \frac{\rho(\rho+1)b_n}{2}\Phi_i^n = \frac{\rho(\rho+1)}{2}{\tau_n}^2\sum f + O({\tau_n}^4 + {\tau_n}^2h^2).$ (28)

The above linear scheme is conditionally stable even for uniform mesh cases in both directions. To find an unconditionally stable scheme of the same accuracy, we may re-write (28) into a similar form

$\left[1 + \frac{2(\rho-1)}{3}\sqrt{a_n} + \gamma_1 b_n - \gamma_2{p_n}^2{\delta_x}^2\right]$
$[\Phi_i^{n+1} - (1+\rho)\Phi_i^n + \rho\Phi_i^{n-1}] + \left[\sqrt{a_n} + \frac{(\rho-1)}{6}b_n - \frac{(\rho-1)}{6}{p_n}^2{\delta_x}^2\right][\Phi_i^{n+1} - (1-\rho^2)\Phi_i^n - \rho^2\Phi_i^{n-1}]$
$-\frac{\rho(\rho+1)}{2}{p_n}^2{\delta_x}^2\Phi_i^n + \frac{\rho(\rho+1)b_n}{2}\Phi_i^n = \frac{\rho(\rho+1)}{24}{\tau_n}^2\sum f + O({\tau_n}^4 + {\tau_n}^2h^2),$ (29)

where γ_1, γ_2 are parameters to be determined and the additional term $[\gamma_1 b_n - \gamma_2{p_n}^2{\delta_x}^2][\Phi_i^{n+1} - (1+\rho)\Phi_i^n + \rho\Phi_i^{n-1}]$ is of the higher order, does not alter the accuracy of the scheme.

Assume that there exists an error $\epsilon_i^n = \Phi_i^n - \phi_i^n$ at each internal grid point (x_i, t_n), the corresponding error equation is given by

$\left[1 + \frac{2(\rho-1)}{3}\sqrt{a_n} + \gamma_1 b_n - \gamma_2{p_n}^2{\delta_x}^2\right]$
$[\epsilon_i^{n+1} - (1+\rho)\epsilon_i^n + \rho\epsilon_i^{n-1}] + \left[\sqrt{a_n} + \frac{(\rho-1)}{6}b_n - \frac{(\rho-1)}{6}{p_n}^2{\delta_x}^2\right][\epsilon_i^{n+1} - (1-\rho^2)\epsilon_i^n - \rho^2\epsilon_i^{n-1}]$
$-\frac{\rho(\rho+1)}{2}{p_n}^2{\delta_x}^2\epsilon_i^n + \frac{\rho(\rho+1)b_n}{2}\epsilon_i^n = O({\tau_n}^4 + {\tau_n}^2h^2)$ (30)

For stability interval of the scheme (29), at each mesh point (x_i, t_n), we consider the error of the form $\epsilon_i^n = \xi^n.\exp(\theta i\sqrt{-1})$, where θ is real & ξ is in general a complex number. Thus using $\epsilon_i^n = \xi^n.\exp(\theta i\sqrt{-1})$ in the non-homogeneous part of the error equation (30), we obtain the corresponding characteristic equation

$A_0\xi^2 + B_0\xi + C_0 = 0,$ (31)

Where

$A_0 = \left[1 + \frac{2(\rho-1)}{3}\sqrt{a_n} + \gamma_1 b_n + 4\gamma_2{p_n}^2 sin^2\frac{\theta}{2}\right]$
$+\left[\sqrt{a_n} + \frac{(\rho-1)}{6}b_n + \frac{2(\rho-1)}{3}{p_n}^2 sin^2\frac{\theta}{2}\right],$ (32)

$B_0 = -(1+\rho)\left[1 + \frac{2(\rho-1)}{3}\sqrt{a_n} + \gamma_1 b_n + 4\gamma_2{p_n}^2 sin^2\frac{\theta}{2}\right]$
$-(1-\rho^2)\left[\sqrt{a_n} + \frac{(\rho-1)}{6}b_n + \frac{2(\rho-1)}{3}{p_n}^2 sin^2\frac{\theta}{2}\right]$
$+2\rho(1+\rho){p_n}^2 sin^2\frac{\theta}{2} + \frac{\rho(1+\rho)}{2}b_n,$ (33)

$C_0 = \rho\left[1 + \frac{2(\rho-1)}{3}\sqrt{a_n} + \gamma_1 b_n + 4\gamma_2{p_n}^2 sin^2\frac{\theta}{2}\right]$
$-\rho^2\left[\sqrt{a_n} + \frac{(\rho-1)}{6}b_n + \frac{2(\rho-1)}{3}{p_n}^2 sin^2\frac{\theta}{2}\right].$ (34)

For stability, the necessary and sufficient conditions for $|\xi| < 1$ are that

$A_0 + B_0 + C_0 > 0,\ A_0 - C_0 > 0$ and $A_0 - B_0 + C_0 > 0.$

The condition $A_0 + B_0 + C_0 = 2\rho(1+\rho){p_n}^2 sin^2\frac{\theta}{2} + \frac{\rho(1+\rho)}{2}b_n > 0,$ (35)

is fulfilled for $\alpha_0 > 0$, $\beta_0 \geq 0$ and for all θ apart from $\theta = 0$ or 2π and $\beta_0 = 0$.

We will consider this case at the end of this section.

The condition

$A_0 - C_0 = (1-\rho)[1 + (\gamma_1 - \frac{1+\rho^2}{6})b_n$
$+4\left(\gamma_2 - \frac{1+\rho^2}{6}\right){p_n}^2 sin^2\frac{\theta}{2}] + \frac{(1+4\rho+\rho^2)}{3}\sqrt{a_n} > 0,$ (36)

must be satisfied for all $\alpha_0 > 0$, $\beta_0 \geq 0$ provided $0 < \rho \leq 1, \gamma_1 \geq \frac{1+\rho^2}{6}, \gamma_2 \geq \frac{1+\rho^2}{6}$.

Finally, the condition

$A_0 - B_0 + C_0 = 2(1+\rho)\left[1 + \frac{(1-\rho)}{3}\sqrt{a_n}\right.$
$\left. + \left(\gamma_1 - \frac{2\rho^2-\rho+2}{12}\right)b_n + 4\left(\gamma_2 - \frac{2\rho^2-\rho+2}{12}\right){p_n}^2 sin^2\frac{\theta}{2}\right] > 0,$ (37)

must be satisfied for all $\alpha_0 > 0$, $\beta_0 \geq 0$ provided $0 < \rho \leq 1, \gamma_1 \geq \frac{2-\rho+2\rho^2}{12}, \gamma_2 \geq \frac{2-\rho+2\rho^2}{12}$.

When $\theta = 0$ or 2π and $\beta_0 = 0$, the characteristic equation (31) becomes

$$\left[1+\left(\frac{1+2\rho}{3}\right)\sqrt{a_n}\right]\xi^2 - \left[(1+\rho)\left(1+\frac{2(\rho-1)}{3}\sqrt{a_n}\right)+(1-\rho^2)\sqrt{a_n}\right]\xi + \rho\left[1+\frac{2(\rho-1)}{3}\sqrt{a_n}-\rho\sqrt{a_n}\right]=0. \quad (38)$$

If ξ_1 and ξ_2 are two roots of (38), we have the relations

$$\xi_1+\xi_2 = 1+\frac{\rho\left[1-\frac{(\rho+2)}{3}\sqrt{a_n}\right]}{1+\frac{(1+2\rho)}{3}\sqrt{a_n}}, \quad (39)$$

$$\xi_1 . \xi_2 = \frac{\rho\left[1-\frac{(\rho+2)}{3}\sqrt{a_n}\right]}{1+\frac{(1+2\rho)}{3}\sqrt{a_n}}. \quad (40)$$

Solving equations (39) and (40), we get

$$\xi_1 = 1,\ \xi_2 = \frac{\rho\left[1-\frac{(\rho+2)}{3}\sqrt{a_n}\right]}{1+\frac{(1+2\rho)}{3}\sqrt{a_n}}.$$

In this case also $|\xi| \le 1$, provided $0<\rho\le 1$.

Since $\frac{1+\rho^2}{6} > \frac{2-\rho+2\rho^2}{12}$, the conditions (35)-(37) are satisfied for all variable angle θ, $\alpha_0 > 0$, $\beta_0 \ge 0$ provided $0<\rho\le 1$, $\gamma_1 \ge \frac{1+\rho^2}{6}, \gamma_2 \ge \frac{1+\rho^2}{6}$. Thus for $\alpha_0 > 0$, $\beta_0 \ge 0$, $0<\rho\le 1, \gamma_1 \ge \frac{1+\rho^2}{6}, \gamma_2 \ge \frac{1+\rho^2}{6}$, the scheme (29) is stable for all possibilities of $\tau_n > 0$ and $h > 0$.

5. Computational results

With the help of the approximations (14), (15) and (17), a variable mesh method of $O(\tau_n + h^2)$ in the t-direction can be written as

$$\bar{\Phi}_{tt_i}^{\ n} - \alpha_i^n \bar{\Phi}_{xx_i}^{\ n} = \bar{R}_i^n + O(\tau_n + h^2). \quad (41)$$

We have solved several standard problems arising from physics and engineering using the method (20) and compared our values with those achieved by employing the scheme (41). The analytical solutions are known in each case. We determine the right-hand side homogeneous function, initial & boundary conditions using the analytical solution as a test process. The tri-diagonal solver can be employed for solving the linear difference equation and Newton-Raphson method for non-linear difference equations (Hageman *et al.* (2004)). We use MATLAB codes to perform all the numerical computations.

The derived methods (20), (29) and (41) for second order HPDEs are three-level implicit schemes. To commence the estimation, it's mandatory to calculate the approximate solution of ϕ of desired accuracy at $t=\tau_1$.As it is given the values of ϕ and ϕ_t at t = 0 explicitly, we can determine the values of subsequent tangential derivatives of ϕ and ϕ_t at t = 0, which implies that at t = 0, the values of $\phi, \phi_x, \phi_{xx} ..., \phi_t, \phi_{tx}, \phi_{txx}$, etc are known.

Using Taylor's expansion, an approximation at first time level is given by

$$\Phi_i^1 = \Phi_i^0 + \tau_1(\Phi_t)_i^0 + O({\tau_1}^2). \quad (42)$$

From Eq. (1), we have

$$(\Phi_{tt})_i^0 = [\alpha(x,t,\Phi)\Phi_{xx} + R(x,t,\Phi,\Phi_x,\Phi_t]_i^0. \quad (43)$$

Now using the initial values Φ and its derivatives, from (42), we can estimate Φ_{tt} at t = 0 and automatically, we get the numerical value of Φ of required accuracy at first time level, that is, at $t=\tau_1$.

Example 1. We solve the Telegraph equation (25) in the solution domain $0 < x < 1$, $t > 0$. The analytical solution is given by $\phi(x,t) = \exp(-2t)\,sinhx$. The maximum absolute errors (MAEs) are reported in Table 1 at $t=1$ for different values of $\alpha_0, \beta_0, \gamma_1$and γ_2 for $\eta = 0.9$and $\eta = 1$. Figure 1 shows the numerical vs. exact solution at $t = 1$ for $\alpha_0 = 10$, $\beta_0 = 5, \gamma_1 = 0.5, \gamma_2 = 1$and $\eta = 0.9$.

Table 1. The maximum absolute errors (using proposed method) of Example 1 at $t=1.0$ for $\gamma = 1.0$. (with CPU time in seconds)

N	$\eta = 0.9$		$\eta = 1$
	$\alpha = 10, \beta = 5,$ $\gamma_1 = 0.5, \gamma_2 = 1.0$		$\alpha = 10, \beta = 5,$ $\gamma_1 = 0.5, \gamma_2 = 1.0$
	Scheme (29)	Scheme (41)	Scheme (20)
16 (CPU time)	5.0606e-04 (0.036507)	5.9262e-04 (0.027388)	1.6882e-05 (0.028633)
32 (CPU time)	5.0905e-04 (0.082532)	5.9524e-04 (0.05804)	4.0745e-06 (0.075220)
64 (CPU time)	5.0947e-04 (0.369452)	5.9601e-04 (0.336036)	9.5492e-07 (0.115225)

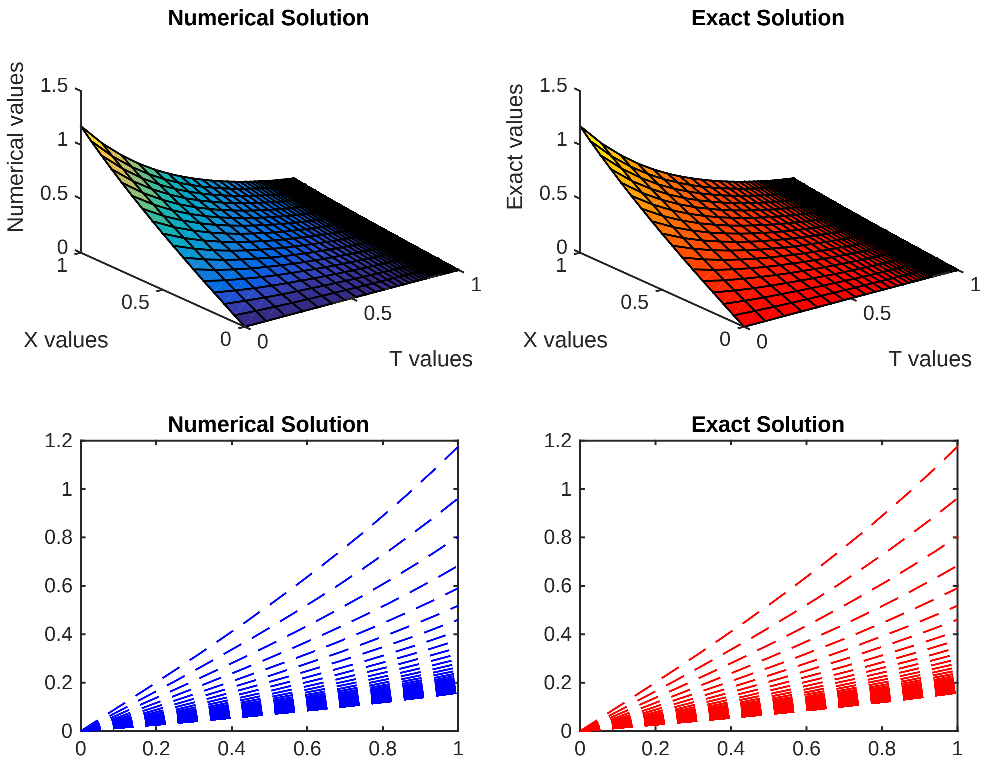

Figure 1. The graph of numerical vs. exact solution of example 1 at $t = 1, \eta = 0.9, \gamma_1 = 0.5, \gamma_2 = 1.0, N = 16$.

Example 2. Wave equation in polar coordinates

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{\gamma}{r}\frac{\partial \phi}{\partial r} + f(r,t), 0 < r < 1, t > 0 \qquad (44)$$

We solve (44) using the technique discussed in Mohanty *et al.* (1996). The analytical solution is specified by $\phi(r,t) = r^2 sinht$. The MAEs are reported in Table 2 at t =1 for $\gamma = 1$, $\eta = 1.02$ and $\eta = 1$. The numerical vs. exact solution curves are plotted in Figure 2 at t = 1 for γ = 1 and $\eta = 1.02$.

Table 2. The maximum absolute errors of Example 2 at $t = 1.0$ (with CPU time in seconds) for $\gamma = 1.0$

N	$\eta = 1.02$		$\eta = 1$
	Scheme (20)	Scheme (41)	Scheme (20)
16 (CPU time)	1.2395e-06 (0.007265)	1.1234e-05 (0.006221)	1.9710e-07 (0.010309)
32 (CPU time)	6.7185e-07 (0.032558)	7.4206e-06 (0.021260)	4.7481e-08 (0.01882)
64 (CPU time)	6.6952e-07 (0.088161)	7.4105e-06 (0.078849)	1.1450e-08 (0.097452)

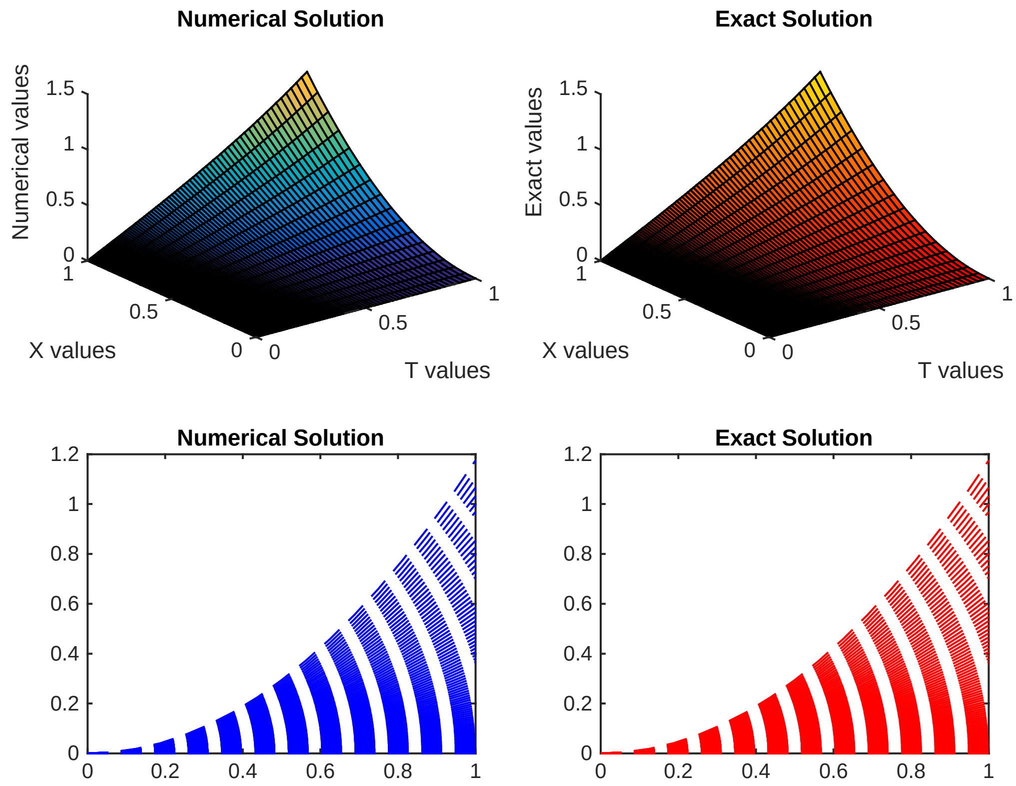

Figure 2. The graph of numerical vs. exact solution of example 2 at $= 1, \eta = 1.02, \gamma = 1$,$N = 16$.

Example 3. Vander Pol type non-linear wave equation

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} + \gamma(\phi^2 - 1)\frac{\partial\phi}{\partial t} + f(x,t),\ 0 < x < 1,\ t > 0 \quad (45)$$

The analytical solution is given by $\phi(x,t) = \exp(-\gamma t)\sin(\pi x)$. The MAEs are reported in Table 3 at t =1 for $\gamma = 1$, $\eta = 0.9$ and $\eta = 1$. The numerical vs. exact solution curves are plotted in Figure 3 at t = 1 for γ = 1 and $\eta = 1.02$.

Table 3. The maximum absolute errors for Example 3 at $t = 1.0$ (with CPU time in seconds) for $\gamma = 1.0$

N	$\eta = 1.02$		$\eta = 1$
	Scheme (20)	Scheme (41)	Scheme (20)
16 (CPU time)	3.2018e-03 (0.012291)	3.2205e-03 (0.013780)	3.2021e-03 (0.010624)
32 (CPU time)	7.9884e-04 (0.032748)	8.0577e-04 (0.025513)	7.9855e-04 (0.027632)
64 (CPU time)	2.0005e-04 (0.113814)	2.0629e-04 (0.118188)	1.9950e-04 (0.121405)

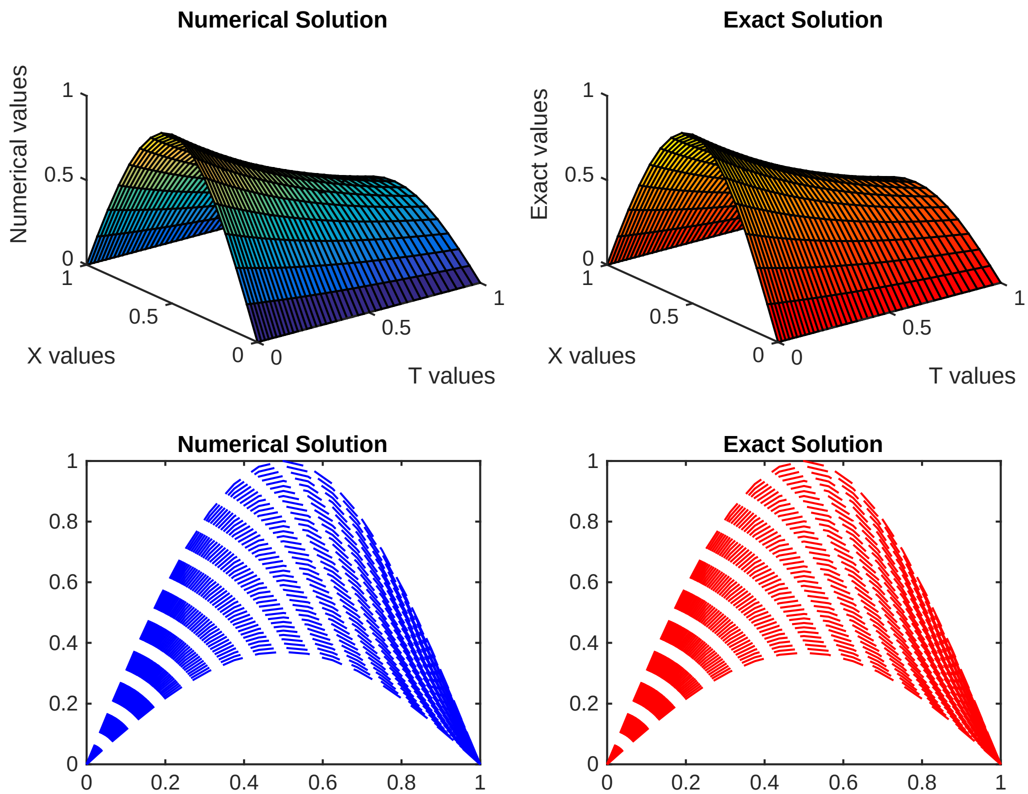

Figure 3. The graph of numerical vs. exact solution of example 3 at $= 1, \eta = 1.02, \gamma = 1$,$N = 16$.

Example 4. Dissipative non-linear wave equation

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} - 2\phi\frac{\partial\phi}{\partial t} + f(x,t), \qquad 0 < x < 1,\ t > 0 \qquad (46)$$

The analytical solution is defined by $\phi(x,t) = \sin(\pi x)\cosh(t)$. The MAEs are reported in Table 4 at t =1 for $\eta = 1.02$ and $\eta = 1$. The numerical vs. exact solution curves are displayed in Figure 4 at t = 1 for $\eta = 1.02$.

Table 4. The maximum absolute errors for Example 4 at $t = 1.0$ (with CPU time in seconds)

N	$\eta = 1.02$		$\eta = 1$
	Scheme (20)	Scheme (41)	Scheme (20)
16 (CPU time)	4.5962e-03 (0.015540)	4.6089e-03 (0.025324)	4.6163e-03 (0.010389)
32 (CPU time)	1.1533e-03 (0.046219)	1.1400e-03 (0.074779)	1.1590e-03 (0.026104)
64 (CPU time)	2.7891e-04 (0.135696)	2.8400e-04 (0.094694)	2.9006e-04 (0.095441)

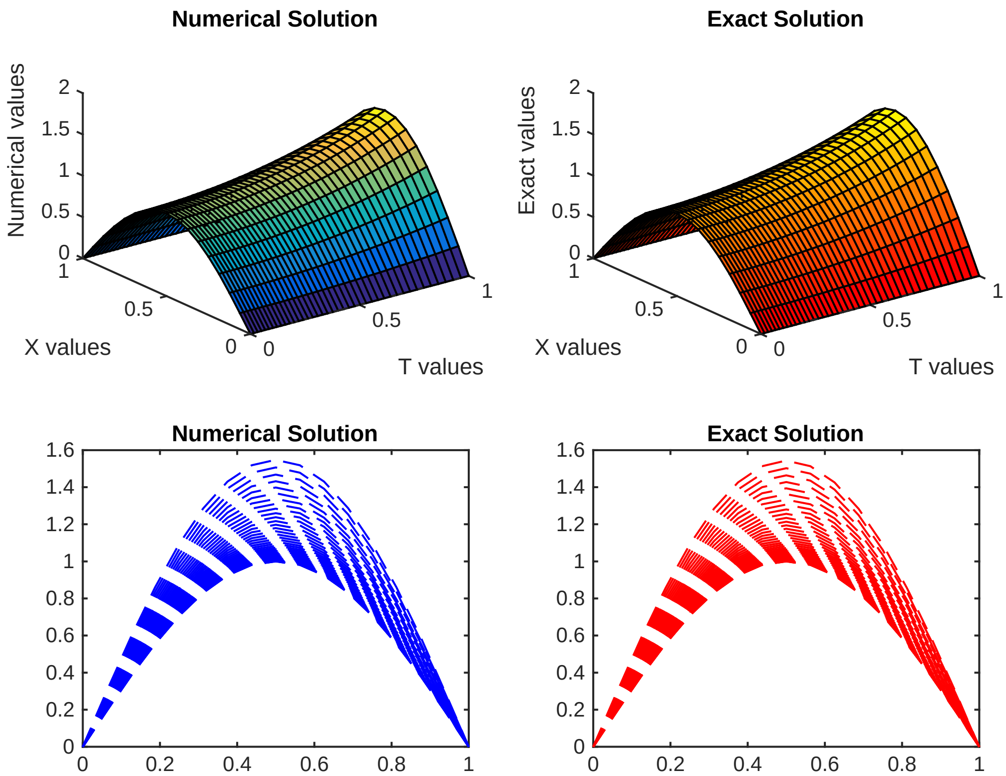

Figure 4. The graph of numerical vs. exact solution of Example 4 at $= 1, \eta = 1.02$,$N = 16$.

***Example* 5:** Quasi-linear equation

$$\frac{\partial^2\phi}{\partial t^2} = (1+\phi^2)\frac{\partial^2\phi}{\partial x^2} + \phi\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t}\right) + f(x,t), \quad 0 < x < 1,\ t > 0 \tag{47}$$

The analytical solution is given by $\phi(x,t) = \exp(-2t)\sin(\pi x)$. The MAEs at t =1 for $\eta = 1.02$ and $\eta = 1$ are reported in Table 5. The numerical vs. exact solution curves are displayed in Figure 5 at t = 1 for $\eta = 1.02$.

Table 5. The maximum absolute errors of Example 5 for $t = 1.0$ (with CPU time in seconds)

N	$\eta = 1.02$		$\eta = 1$
	Scheme (20)	Scheme (41)	Scheme (20)
16 (CPU time)	2.7465e-03 (0.015523)	9.7652e-03 (0.012586)	2.6699e-03 (0.012885)
32 (CPU time)	7.2357e-04 (0.065103)	6.9598e-04 (0.020543)	6.7040e-04 (0.024150)
64 (CPU time)	2.1534e-04 (0.124379)	1.9434e-04 (0.096065)	1.6747e-04 (0.102078)

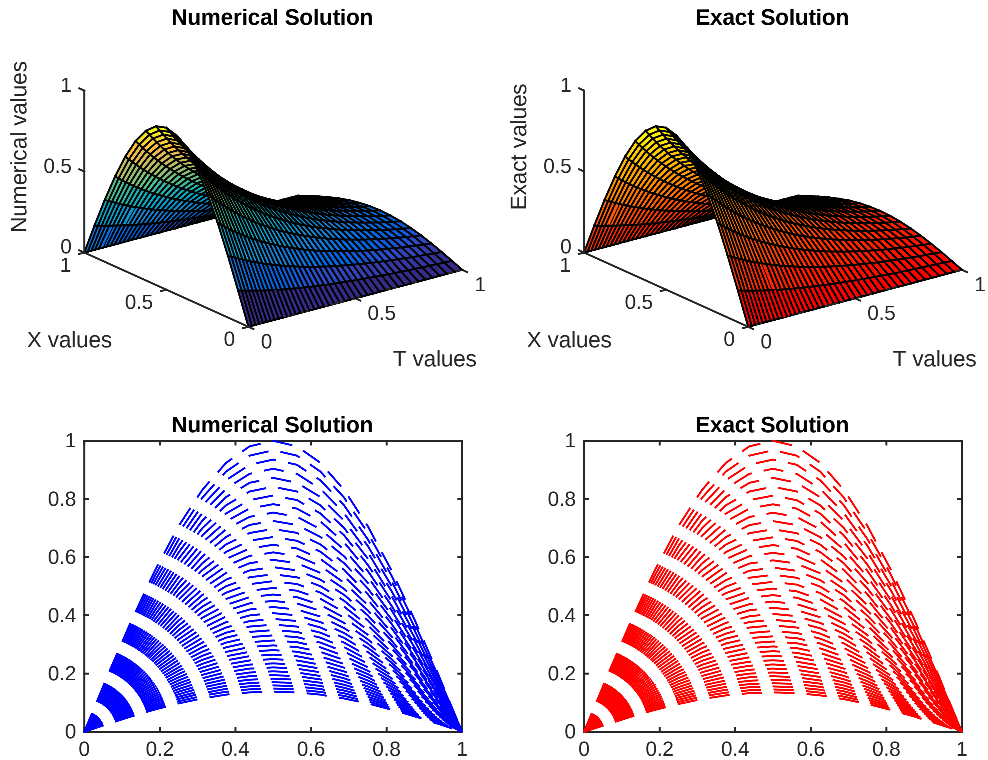

Figure 5. The graph of numerical vs. exact solution of Example 5 at $= 1, \eta = 1.02$, $N = 16$.

Employing the following formula, we calculate the order of convergence of the scheme for uniform case i.e. for $=1.0$:

$$\frac{\log(e_{h_1})-\log(e_{h_2})}{\log(h_1)-\log(h_2)} \quad (48)$$

where maximum absolute errors are e_{h_1} and e_{h_2} for two consecutive uniform mesh sizes h_1 and h_2 respectively.

To calculate the order of convergence of the suggested method, MAEs for the last two values of h, i.e., $h_1 = 1/32$ and $h_2 = 1/64$ have been considered, and corresponding results are presented in Table 6.

Table 6. Order of the convergence

Example	Parameters	Order of the method
1	$\alpha = 10, \beta = 5.0, \gamma_1 = 0.5,$ $\gamma_2 = 1.0$ at $t = 1$	2.0932
2	at $t = 1.0, \gamma = 1.0$	2.0520
3	at $t = 1.0, \gamma = 1.0$	2.0010
4	at $t = 1.0$	1.9985
5	at $t = 1.0$	2.0011

6. Conclusion

In this paper, using three level variable mesh points in the t-direction and three uniform mesh points in space direction, we have discussed a novel stable numerical technique of $O({\tau_n}^2 + h^2)$ for the numerical solution of 1D quasi-linear HPDEs (1.1). The proposed method, when applied to the Telegraph equation, is shown to be unconditionally stable on a variable grid point in the t-

direction, and the stability criterion is established. We have solved some noteworthy problems on linear and non-linear wave equations to justify the usefulness of the proposed methods. The accuracy and efficacy of the developed methods are demonstrated from the computed numerical results. Furthermore, the suggested technique can be extended to 2D and 3D quasi-linear second order hyperbolic partial differential equations on a graded mesh in the time direction.

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