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A Novel Numerical Scheme for 1D Parabolic Equation on a Graded Mesh: Application to Burgers' Equation

Research Article

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Abstract: We develop a two level compact implicit finite difference method of order two on a graded mesh for the approximation of 1D nonlinear parabolic equation $\epsilon w_{xx} = \psi(x, t, w, w_x, w_t)$, $(x, t) \in \Omega = [a, b] \times [0, T]$. The proposed method is unconditionally stable when it is applied to the linear equation. The significant advantage of this method is that, it is directly pertinent to singular parabolic equations to obtain oscillation-free solutions. To validate the applicability of the proposed method, two model examples are considered and solved for different values of mesh sizes in both directions. The convergence has been shown in the sense of maximum absolute error. The proposed method is validated via the same numerical test examples. The present method approximates exact solution very well. The linear difference equations have been solved using a tri-diagonal solver, whereas Newton–Raphson method have been used to solve non-linear difference equations.

Keywords: *Graded mesh • Singular problem • Burgers • Equation • Burgers–Huxley equation*

1. Introduction

Nonlinear parabolic partial differential equations have been extensively studied as they encounter in several areas of mathematical, physical and engineering sciences, particularly in physics, biology, finance and chemistry. These type of PDEs are used to interpret various phenomena such as viscous fluid flow, heat conduction and transfer, filtration of liquids, chemical reactions, water transfer in solids, dispersion of traces in porous media, environmental pollution etc. (Mittal *et al.* 2011, Zhou *et al.* 2011, Jain *et al.* 1990). During last few decades, the problem of finding numerical solution of quasi linear parabolic PDEs attained much attention of researchers due to their practical importance. In (Mittal *et al.* 2011, Liu *et al.* 2011, Dehghan *et al.* 2014, Jain *et al.* 1990, Mohanty *et al.* 2007), many researchers studied several parabolic PDEs of different physical models and developed numerical schemes based on approaches like finite difference, finite element, finite volume, spline and many more. In this article, we deem the following general form of nonlinear parabolic partial differential equations (PDEs)

$$\epsilon w_{xx} = \psi(x, t, w, w_x, w_t), \ (x, t) \in \Omega = [a, b] \times [0, T] \quad (1)$$

subject to the initial condition

$$w(x, 0) = f_0(x), \qquad a \le x \le b \quad (2)$$

and prescribed boundary conditions

$$w(a, t) = f_1(x), \quad w(b, t) = f_2(t), \quad t > 0 \quad (3)$$

We presume that the functions $\psi(x, t, w, w_x, w_t), f_r(x)$, $r = 0, 1, 2$ are adequately smooth and their required higher order derivatives exist in solution region Ω.

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Some particular forms of equation (1) are very well known differential equations, namely, Burgers', Burgers–Huxley, Burgers–Fishers.

Burger's equation has broad range of applications in various fields e.g.: fluid and gas dynamics, nonlinear acoustics, elasticity etc. and it is generally corresponding to Navier–Stoke's equations (Mittal *et al.* 2012, Kadalbajoo *et al.* 2006). The Burgers'–Fishers equation arises in heat conduction and plasma physics, fluid mechanics and gas dynamics (Brazhnik *et al.* 1999, Mohammadi 2011, Zhu *et al.* 2010). Burgers'–Huxley equation shows a sample model for relating the interface between reaction mechanisms and diffusion transports convection (Macías-Díaz 2018, Celik 2016, Duan *et al.* 2012, Wang *et al.* 1990). Therefore, many eminent nonlinear reaction–diffusion problems can be constructed by equations (1)-(3).

Beforehand, it is worth mentioning that there is a huge number of different approaches existing in the literature to approximate the solutions of equations (1)-(3) and its particular forms. For instance, several authors have introduced different numerical methods in (Mohanty 2007, Jain *et al.* 2009). Recently, there is a huge number of works in the literature on parabolic equations such as Alper Korkmaza *et al.* (2011) studied differential quadrature algorithm based on Lagrange interpolation polynomials for the approximation of nonlinear Burgers' equation. Tomasiello (2010) worked on Burgers–Huxley equation and this equation was studied numerically by Iterative Differential Quadrature (IDQ) method which is based on quadrature rules. Athanassios G. Bratsos (2011) proposed a two-level finite difference scheme, having accuracy of order four, for the numerical solution of the generalized Burger-Hauxly equation. Zhang *et al.* (2012) presented a numerical method by applying the local irregular Galerkin method to solve the generalized Burgers–Huxley equation and the generalized Burgers–Fisher equation numerically. Mohebbi *et al.* (2010) projected a class of fresh finite difference techniques for solving the 1D heat and advection–diffusion equations. Goh *et al.* (2012) obtained approximate solutions of 1D heat and advection-diffusion equations by collocation method based on cubic B-spline. Mohanty, *et al.* (2017) used Spline in compression approximations along with half-step mesh for the solution of the system of 1D quasi-linear PDEs. Ghasemi (2018) presented a numerical method for the solution of Nonlinear Parabolic Equations using extrapolated collocation method. Above methods cannot be applied directly to solve singular parabolic PDEs. Most recently Ghosh *et al.*(2021) eshtablished a method for the solution of 1D hyperbolic equation using graded mesh in time direction. In this present paper, we developed a two level compact implicit finite difference scheme in exponential form by using graded mesh for the solution of nonlinear PDEs (1)-(3).

The paper is arranged as follows:

Section. 2 covers the formulation of numerical scheme. Sec. 3 describes the complete derivation of the scheme. Sec. 4 covers the detailed stability analysis. In Sec. 5, several yardstick examples are solved to justify the method's efficacy. At last, Sec. 6 presents the concluding notes.

2. Conceptualization of the variable mesh method

Solution domain Ω of equations (1)-(3) is discretized by graded mesh in spatial direction such that $a = x_0 < x_1 < \cdots < x_K < x_{K+1} = b$, where K is the number of meshes. Let the graded mesh size be $h_i = x_i - x_{i-1}$, for $i = 1, 2, \ldots, K+1$ then mesh ratio is defined as $\eta_i = \frac{h_{i+1}}{h_i} > 0$.

Node points are given by $x_i = x_0 + \sum_{r=1}^{i} h_r$, where $i = 1,2,\ldots,K+1$. We use constant mesh in time direction defined as $k = t_n - t_{n-1} > 0$ for $= 1,2,\ldots$, where t_n is node point in time direction.

At grid point (x_i, t_n), let $W_i^n = w(x_i, t_n)$ denotes the exact solution and w_i^n denotes approximate solution of equations (1)-(3). For simplicity, we consider $\eta_i = \eta$ (a constant $\neq 1$), $i = 1(1)K+1$. For $\eta = 1$ discretization reduces to the uniform mesh case. For η >1 or η <1, the mesh sizes are either increasing or decreasing in order.

In the term of approximate value w_i^n at the grid point (x_i, t_n), the above defined differential equation (1) may be written as

$$\epsilon {w_{xx}}_i^n = \psi(x_i, t_n, w_i^n, {w_x}_i^n, {w_t}_i^n) \equiv \Psi_i^n. \tag{4}$$

We denote some constants by

$$\mathrm{P} = \eta^2 + \eta - 1,$$
$$\mathrm{Q} = (1+\eta)(1+3\eta + \eta^2),$$
$$\mathrm{R} = \eta(1+\eta - \eta^2),$$
$$\mathrm{S} = \eta(1+\eta).$$

We need the following approximations

$$\bar{t_n} = t_n + \theta k, \tag{5}$$
$$\bar{W}_i^n = \theta W_i^{n+1} + (1-\theta) \mathrm{W}_i^n, \tag{6}$$
$$\bar{W}_{i+1}^n = \theta \mathrm{W}_{i+1,}^{n+1} + (1-\theta) W_{i+1,}^n, \tag{7}$$
$$\bar{\mathrm{W}}_{i-1}^n = \theta \mathrm{W}_{i-1,}^{n+1} + (1-\theta) \mathrm{W}_{i-1,}^n \tag{8}$$
$$\bar{\mathrm{W}}_{t_i}^{\ n} = \frac{1}{k}\left(\mathrm{W}_i^{n+1} - \mathrm{W}_i^n\right), \tag{9}$$
$$\bar{\mathrm{W}}_{t_{i+1}}^{\ n} = \frac{1}{k}\left(\mathrm{W}_{i+1}^{n+1} - \mathrm{W}_{i+1}^n\right), \tag{10}$$
$$\bar{\mathrm{W}}_{t_{i-1}}^{\ n} = \frac{1}{\tau}\left(\mathrm{W}_{i-1}^{n+1} - \mathrm{W}_{i-1}^n\right), \tag{11}$$

$$\bar{W}_{x_i}^{\ n} = (\bar{W}_{i+1}^n - (1-\eta^2)W\bar{U}_i^n - \eta^2\bar{W}_{i-1}^n)/\eta(1+\eta)h_{i,} \quad (12)$$

$$\bar{W}_{x_{i+1}}^{\ n} = ((1+2\eta)\bar{W}_{i+1}^n - (1+\eta)^2\bar{W}_i^n + \eta^2\bar{W}_{i-1}^n)/\eta(1+\eta)h_{i,} \quad (13)$$

$$\bar{W}_{x_{i-1}}^{\ n} = (-\bar{W}_{i+1}^n + (1+\eta)^2\bar{W}_i^n - \eta(2+\eta)\bar{W}_{i-1}^n)/\eta(1+\eta)h_{i,} \quad (14)$$

$$\bar{\Psi}_{i+1}^n = \psi\left(x_{i+1}, \bar{t}_n, \bar{W}_{i+1}^{\mathrm{n}}, \bar{W}_{x_{i+1}}^{\ \mathrm{n}}, \bar{W}_{t_{i+1}}^{\ \mathrm{n}}\right), \quad (15)$$

$$\bar{\Psi}_{i-1}^n = \psi\left(x_{i-1}, \bar{t}_n, \bar{W}_{i-1}^{\mathrm{n}}, \bar{W}_{x_{i-1}}^{\ \mathrm{n}}, \bar{W}_{t_{i-1}}^{\ \mathrm{n}}\right), \quad (16)$$

$$\bar{\bar{W}}_{x_i}^{\ n} = \bar{W}_{x_i}^{\ n} + ah_i[\bar{\Psi}_{i+1}^n - \bar{\Psi}_{i-1}^n], \quad (17)$$

$$\bar{\bar{\Psi}}_i^j = \Psi\left(x_i, \bar{t}_n, \bar{W}_i^{\mathrm{n}}, \bar{\bar{W}}_{x_i}^{\ \mathrm{n}}, \bar{W}_{t_i}^{\ \mathrm{n}}\right), \quad (18)$$

$$\hat{\bar{W}}_{x_i}^{\ n} = \bar{W}_{x_i}^{\ n} + bh_i[\bar{\Psi}_{i+1}^n - \bar{\Psi}_{i-1}^n], \quad (19)$$

$$\hat{\bar{\Psi}}_i^j = \Psi\left(x_i, \bar{t}_n, \bar{W}_i^{\mathrm{n}}, \hat{\bar{W}}_{x_i}^{\ \mathrm{n}}, \bar{W}_{t_i}^{\ \mathrm{n}}\right). \quad (20)$$

where a and b are the parameters to be resoluted.

Then at every internal grid point (x_i, t_n), the Eq. (1) is discretized by

$$\bar{W}_{i+1}^n - (1+\eta)\bar{W}_i^n + \eta\bar{W}_{i-1}^n = \eta(1+\eta)\frac{h_i^2}{2}\bar{\bar{\Psi}}_i^n \exp\left[\frac{P\bar{\Psi}_{i+1}^n + R\bar{\Psi}_{i-1}^n - (P+R)\hat{\bar{\Psi}}_i^n}{6\eta(1+\eta)\bar{\bar{\Psi}}_i^j}\right] + \hat{T}_i^n \quad (21)$$

where $\theta = \frac{1}{2}$, $a = -\frac{\eta}{6(1+\eta)}$,

$b = -\frac{\eta(1-5\eta+\eta^2)}{6(1+\eta)(1-3\eta+\eta^2)}$, $\hat{T}_i^n = O(k^2h_i^2 + kh_i^3 + h_i^5)$.

3. Derivation of the scheme using variable mesh

In this section we talk about the complete derivation of numerical scheme.

At the grid point (x_i, t_n), we denote

$$\Phi_{cd} = \frac{\partial^{c+d}\Phi_i^n}{\partial x^c\, \partial t^d}, \quad c, d = 0, 1, 2, \ldots, \quad (22)$$

$$\alpha_1 = \Psi_{t_i}^{\ n}, \quad \alpha_2 = \Psi_{\phi_i}^{\ n}, \quad \alpha_3 = \Psi_{\phi_{x_i}}^{\ n}, \quad \alpha_4 = \Psi_{\phi_{t_i}}^{\ n}. \quad (23)$$

Differentiating the Eq. (1) w. r. to t and by the help of (22) and (23), we obtain a relation

$$\epsilon\Phi_{21} = \alpha_1 + \alpha_2\Phi_{01} + \alpha_3\Phi_{11} + \alpha_4\Phi_{02}. \quad (24)$$

By applying the fourth-order compact scheme to the second derivatives in (4) and applying some algebraic manipulations, we can write (4) as

$$W_{i+1}^n - (1+\eta)W_i^n + \eta W_{i-1}^n = \eta(1+\eta)\frac{h_i^2}{2}\Psi_i^n \exp\left[\frac{P\Psi_{i+1}^n + R\Psi_{i-1}^n - (P+R)\Psi_i^n}{6\eta(1+\eta)\Psi_i^n}\right] + O\left(h_i^5\right). \quad (25)$$

Simplifying the approximations from (5) to (16), we obtain

$$\bar{W}_i^n = W_i^n + \theta k W_{t_i}^{\ n} + O(k^2), \quad (26)$$

$$\bar{W}_{i+1}^n = W_{i+1}^n + \theta k W_{t_i}^{\ n} + \theta\eta k h_i W_{xt_i}^{\ n} + O(k^2), \quad (27)$$

$$\bar{W}_{i-1}^n = W_{i-1}^n + \theta k W_{t_i}^{\ n} - \theta k h_i W_{xt_i}^{\ n} + O(k^2), \quad (28)$$

$$\bar{W}_{t_i}^{\ n} = W_{t_i}^{\ n} + \frac{k}{2}W_{tt_i}^{\ n} + O(k^2), \quad (29)$$

$$\bar{W}_{t_{i+1}}^{\ n} = W_{t_{i+1}}^{\ n} + \frac{\eta k h_i}{2}W_{xtt_i}^{\ n} + O(k^2), \quad (30)$$

$$\bar{W}_{t_{i-1}}^{\ n} = W_{t_{i-1}}^{\ n} - \frac{k h_i}{2}W_{xtt_i}^{\ n} + O(k^2), \quad (31)$$

$$\bar{W}_{x_i}^{\ n} = W_{x_i}^{\ n} + \frac{\eta {h_i}^2}{6}W_{xxx_i}^{\ n} + \theta k W_{xt_i}^{\ n} + O\left(kh_i + {h_i}^3\right), \quad (32)$$

$$\bar{W}_{x_{i+1}}^{\ n} = W_{x_{i+1}}^{\ n} - \frac{\eta(1+\eta){h_i}^2}{6}W_{xxx_i}^{\ n} + \theta k W_{xt_i}^{\ n} + O\left(kh_i + {h_i}^3\right), \quad (33)$$

$$\bar{W}_{x_{i-1}}^{\ n} = W_{x_{i-1}}^{\ n} - \frac{(1+\eta){h_i}^2}{6}W_{xxx_i}^{\ n} + \theta k W_{xt_i}^{\ n} + O\left(kh_i + {h_i}^3\right), \quad (34)$$

$$\bar{\Psi}_{i+1}^j = \Psi_{i+1}^j + \theta k(\alpha_1 + \alpha_2 W_{01} + \alpha_3 W_{11}) + \frac{k}{2}\alpha_4 W_{02} - \frac{\eta(1+\eta){h_i}^2}{6}W_{xxx_i}^{\ n}\gamma + O\left(k^2 + kh_i + {h_i}^3\right), \quad (35)$$

$$\bar{\Psi}_{i-1}^j = \Psi_{i-1}^j + \theta k(\alpha_1 + \alpha_2 W_{01} + \alpha_3 W_{11}) + \frac{k}{2}\alpha_4 - \frac{(1+\eta){h_i}^2}{6}W_{xxx_i}^{\ n}\gamma + O\left(k^2 + kh_i + {h_i}^3\right). \quad (36)$$

By using previously introduced approximations (32), (35) and (36) in the equation (17) we obtain

$$\bar{\bar{W}}_{x_i}^{\ n} = W_{x_i}^{\ n} + \theta k W_{xt_i}^{\ n} + O\left(kh_i + {h_i}^3\right), \quad (37)$$

where $a = -\frac{\eta}{6(1+\eta)}$.

Therefore, equation (18) can be written as

$$\bar{\bar{\Psi}}_i^j = \Psi(x_i, \bar{t}_n, \bar{W}_i^{\mathrm{n}}, \bar{\bar{W}}_{x_i}^{\ \mathrm{n}}, \bar{W}_{t_i}^{\ \mathrm{n}}) = \Psi_i^j + \theta k(\alpha_1 + \alpha_2 W_{01} + \alpha_3 W_{11}) + \frac{k}{2}\alpha_4 W_{02} + O\left(k^2 + kh_i + {h_i}^3\right). \quad (38)$$

Similarly, we can rewrite equations (19) and (20) as

$$\hat{\bar{W}}_{x_i}^{\ n} = W_{x_i}^{\ n} + \theta k W_{xt_i}^{\ n} + \frac{{h_i}^2}{6}[\eta + 6b(1+\eta)]W_{xxx_i}^{\ n} + O\left(kh_i + {h_i}^3\right) \quad (39)$$

and

$$\hat{\bar{\Psi}}_i^j = \Psi(x_i, \bar{t}_n, \bar{W}_i^{\mathrm{n}}, \hat{\bar{W}}_{x_i}^{\ \mathrm{n}}, \bar{W}_{t_i}^{\ \mathrm{n}}) = \Psi_i^j + \theta k(\alpha_1 + \alpha_2 W_{01} + \alpha_3 W_{11}) + \frac{k}{2}\alpha_4 + \frac{{h_i}^2}{6}[\eta + 6b(1+\eta)]W_{xxx_i}^{\ n}\alpha_3 + O\left(k^2 + kh_i + {h_i}^3\right). \quad (40)$$

Further, using the approximations (26)–(28), we may write

$$\bar{W}_{i+1}^n - (1+\eta)\bar{W}_i^n + \eta\bar{W}_{i-1}^n = W_{i+1}^n - (1+\eta)W_i^n$$

$$+\eta W_{i-1}^{n} + \theta\frac{\eta(\eta+1)}{2}kh_i^{\,2} + O\big(k^2h_i^{\,2} + kh_i^{\,3} + h_i^{\,5}\big). \quad (41)$$

Now by the help of the relations (35), (36), (38) and (40) from (21) and (25), we get the local truncation error (LTE):

$$\hat{T}_i^n = -\frac{h_i^2}{12}\Bigg[6\eta(\eta+1)k\left(\frac{1}{2}-\theta\right)\alpha_4 - (\eta+1)\frac{h_i^2}{6}\Big(2\eta^2 - (1-3\eta+\eta^2)\big(\eta+6b(1+\eta)\big)\Big)W_{xxx_i}^{\ n}\alpha_3\Bigg] + O\big(k^2h_i^{\,2} + kh_i^{\,3} + h_i^{\,5}\big). \quad (42)$$

The proposed method (21) to be of $O\big(k^2 + kh_i + h_i^{\,3}\big)$, the coefficients of $kh_i^2$and h_i^4 in (42) must be equal to zero. Consequently the values of parameters are $\theta = \frac{1}{2}$,

$b = \frac{-\eta(1-5\eta+\eta^2)}{6(\eta+1)(1-3\eta+\eta^2)}$ and the LTE reduces to

$$\hat{T}_i^n = O(k^2h_i^{\,2} + kh_i^{\,3} + h_i^{\,5}).$$

4. Stability consideration

In this segment, we examine the stability of the scheme for the equation

$$\epsilon w_{xx} = w_t + dw_x,\ (x,t) \in \Omega = [a,b]\times[0,T], \quad (43)$$

where 'd' is a constant. Applying the scheme (21) to the Eq. (43) and neglecting the error term, we obtain the following linear numerical scheme for the solution of the Eq. (43).

$$A_0W_{i+1}^{n+1} + (1-A_0-A_1)W_i^{n+1} + A_1W_{i-1}^{n+1} = B_0W_{i+1}^n + (1-B_0-B_1)W_i^n + B_1W_{i-1,}^n \quad (44)$$

where

$$A_0 = \frac{1}{12} - \frac{R}{12} - \frac{r}{2}\left(1-R-\frac{R^2}{3}\right),$$
$$A_1 = \frac{1}{12} + \frac{R}{12} - \frac{r}{2}\left(1+R+\frac{R^2}{3}\right),$$
$$B_0 = \frac{1}{12} - \frac{R}{12} + \frac{r}{2}\left(1-R+\frac{R^2}{3}\right),$$
$$B_1 = \frac{1}{12} + \frac{R}{12} + \frac{r}{2}\left(1+R+\frac{R^2}{3}\right).$$

Von Neumann linear stability analysis technique is used to detrmine the stability of the linear scheme (44). Let $\varepsilon_k^n = \xi^n e^{i\gamma k}$ is defined as the error at the node (x_k, t_n), where ξ is considered as complex and η as real. Using $\varepsilon_k^n = \xi^n e^{i\gamma k}$ in the Eq. (44), we attain the amplification factor as

$$\xi = \frac{1+rH(\gamma)}{1-rH(\gamma)}, \quad (45)$$

where

$$H(\gamma) = \frac{2(3+R^2)(\cos\gamma - 1) - i6r\sin\gamma}{5+\cos\gamma - iR\sin\gamma}.$$

For stability, it is mandatory that $|\xi|^2 \le 1$. Applying this condition on Eq. (45) yields, Re[$H(\gamma)$]≤ 0 which is the necessary and sufficient condition for linear stability. This condition is fulfilled for all values of R. Thus, it is obvious that the projected scheme when applied to the linear equation is unconditionally stable and generates precise and oscillation-free results for all values of ***R*.**

5. Computational results

We have solved the following problems using the method (21) and compared their results with the method developed by Mohanty (2007). We calculaute the right hand side homogeneous function, initial and boundary conditions from the accurate solutions which are provided in each case. We have used the tri-diagonal solver to solve the linear difference equations and Newton–Raphson technique for non-linear difference equations. All working out were performed by using MATLAB. The domain [0, 1] is divided into (N + 1) points in space-direction with

$0 = x_0 < x_1 < x_2 < x_{3\ldots} < x_N < x_{N+1} = 1, where$

$h_i = x_i - x_{i-1}\ and\ \eta_i = \frac{h_{i+1}}{h_i} > 0,\ i = 1(1)N.$

We may write

$$1 = x_{N+1} - x_0 = (x_{N+1}-x_N) + (x_N - x_{N-1}) + \ldots + (x_1 - x_0) = h_{N+1} + h_N + \cdots + h_1. = (\eta_1 + \eta_1\eta_2 + \cdots + \eta_1\eta_2\eta_3\ldots\eta_N)\,h_1 \quad (46)$$

For effortlessness, we deem $\eta_i = \eta$ (a constant), $= 1(1)N$, then from (46) we have

$$h_1 = \frac{1-\eta}{1-\eta^{N+1}}. \quad (47)$$

Using the Eq. (47), we can calculate the value of h_1 considering the total number of mesh ponts (N + 2), which is the first mesh space at the beginning of the boundary. The remaining mesh is resoluted by $h_{i+1} = \eta h_i,\ ,i = 1(1)N$. Throughout our calculation we utilize the time step

$$k = \frac{1.6}{(N+1)^2}.$$

Problem 1. Burgers' equation

$$\nu\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}, \qquad 0 < x < 1, \qquad t > 0.$$

The analytical solution is specified by

$u(x,t) = \frac{2\nu\pi\sin(\pi x)e^{-\nu\pi^2 t}}{2+\cos(\pi x)e^{-\nu\pi^2 t}}$, where $Re = \nu^{-1} > 0$ is known as the Reynolds number. The root mean square (RMS) errors for u at $t = 1.0$ are presented in Table 1 for a preset η = 0.75 and diverse values of Re. The approximate and analytical solutions of problem 1 are shown in Fig. 1 and Fig. 2 respectively.

Table 1. The RMSs for u at $t = 1.0$

h	Proposed Method			Method developed by Mohanty (2007)		
	$Re = 10^2$	$Re = 10^4$	$Re = 10^6$	$Re = 10^2$	$Re = 10^4$	$Re = 10^6$
$\frac{1}{8}$	1.8738e-04	2.7816e-08	2.7932e-12	1.7200e-04	2.4020e-08	2.412e-12
$\frac{1}{16}$	1.4805e-05	2.2650e-09	2.2751e-13	5.7830e-05	7.7770e-09	7.8040e-13
$\frac{1}{32}$	1.4546e-05	2.2302e-09	2.2402e-13	3.7260e-05	4.9920e-09	5.0090e-13
$\frac{1}{64}$	1.4545e-05	2.2300e-09	2.2400e-13	2.6120e-05	3.4990e-09	3.5180e-13

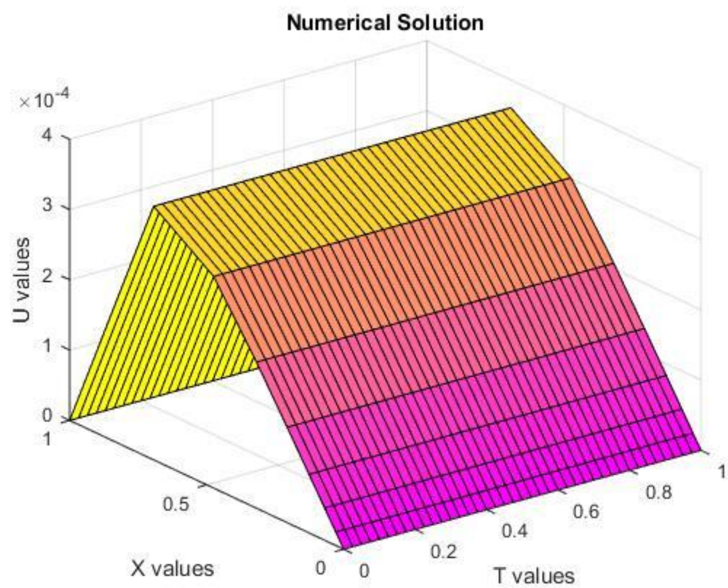

Figure 1. The approximate solution of problem 1 at $t = 1$, for $Re = \nu^{-1} = 10^4$, η = 0.75, $N = 8$.

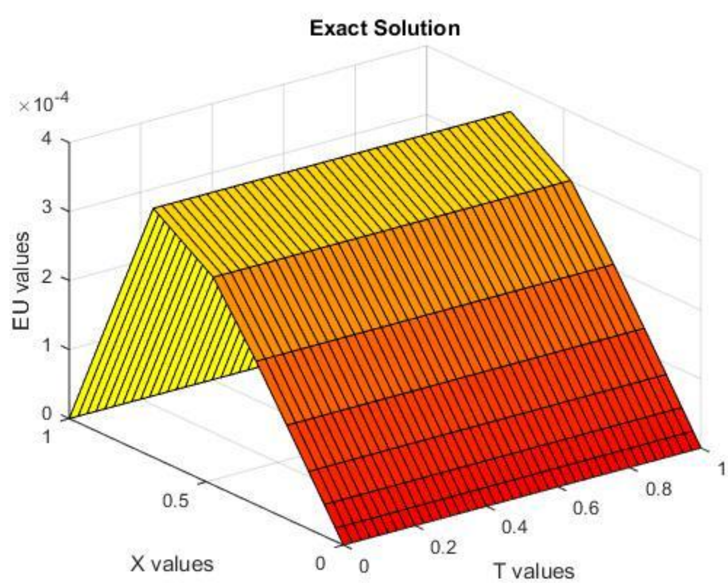

Figure 2. The analytical solution of problem 1 at $t = 1$, for $Re = \nu^{-1} = 10^4$, η = 0.75, $N = 8$.

Problem 2.

$$\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\alpha}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial u}{\partial t}+f(r,t), 0<r<1, t>0.$$

The analytical solution is specified by $u = e^{-\nu t}\cosh r$. The RMS for u at t = 1.0 are presented in Table 2 and 3 for a preset η = 0.8, α = 1 and 2 and diverse values of ν. The approximate and analytical solutions of problem 2 are shown in Fig. 3 and Fig. 4 respectively.

Table 2. The RMS for u at $t = 1.0$

h	Proposed Method			Method developed by Mohanty (2007)		
	$\alpha = 1$			$\alpha = 1$		
	Re = 0.01	*Re* = 0.001	*Re* = 0.0001	*Re* = 0.01	*Re* = 0.001	*Re* = 0.0001
$\frac{1}{8}$	2.1149e-04	2.4203e-04	2.4559e-05	8.8640e-04	1.0520e-04	1.0730e-05
$\frac{1}{16}$	1.8264e-05	2.1223e-05	2.1575e-06	5.7420e-05	7.1560e-05	7.3540e-06
$\frac{1}{32}$	1.8077e-05	2.1030e-05	2.1382e-06	3.9510e-05	4.9720e-05	5.1170e-06
$\frac{1}{64}$	2.5677e-05	1.5601e-05	3.5741e-06	2.7700e-05	3.4870e-05	3.5890e-06

Table 3. The RMS for u at $t = 1.0$

h	Proposed Method			Method developed by Mohanty (2007)		
	$\alpha = 2$			$\alpha = 2$		
	Re = 0.01	*Re* = 0.001	*Re* = 0.0001	*Re* = 0.01	*Re* = 0.001	*Re* = 0.0001
$\frac{1}{8}$	2.1149e-04	2.4203e-04	2.4559e-05	2.2630e-04	3.0780e-04	3.1920e-05
$\frac{1}{16}$	1.8264e-05	2.1223e-05	2.1575e-06	1.3760e-04	2.0440e-05	2.1460e-06
$\frac{1}{32}$	1.8077e-05	2.1030e-05	2.1382e-06	9.3630e-05	1.4140e-05	1.4890e-06
$\frac{1}{64}$	2.5677e-05	1.5601e-06	3.5741e-06	6.5640e-05	9.9240e-06	1.0450e-06

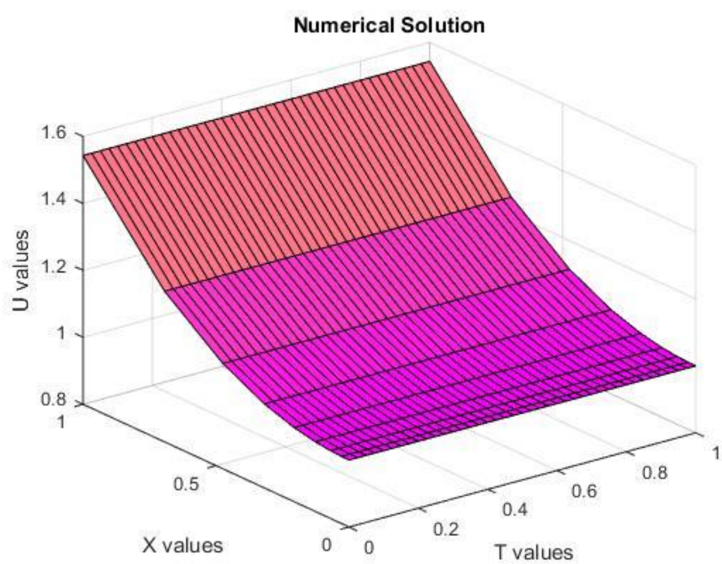

Figure 3. The approximate solution of problem 2 at $t = 1$, for $\alpha = 1$ $Re = 0.001$, η = 0.75, $N = 8$.

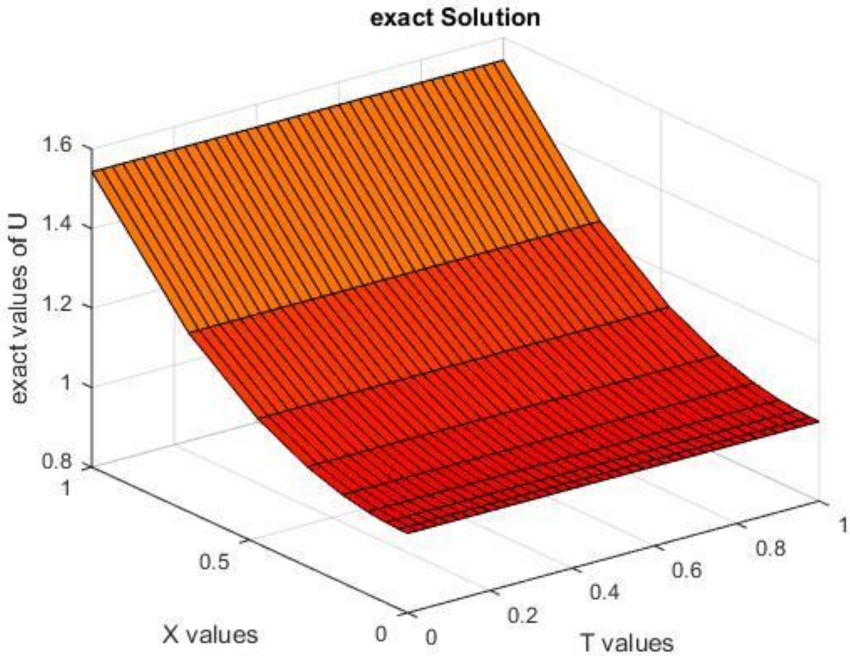

Figure 4. The analytical solution of problem 2 at $t = 1$, for $\alpha = 1$ $Re = 0.001$, η = 0.75, $N = 8$.

6. Conclusion

In this article, we developed a two level compact implicit finite difference scheme in exponential form by using graded mesh for the approximation of nonlinear parabolic PDEs. The most important advantage of the scheme is that, it is directly applicable to singular parabolic PDEs to obtain oscillation-free solutions. The method is compact and the computational stencil requires only nine points at the advanced time level. The proposed method is validated via the graphical and tabular form on the same numerical test examples. The numerical results confirm that the proposed method produce oscillation free solution for large *Re*. At present, we are trying to extend the scheme for more complex flow problems in polar coordinates and the complete Navier-Stokes equations with the pressure being an independent variable.

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