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Energy Estimates of the Signed Solutions to Doubly Nonlinear Parabolic Equations

Research Article

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ABSTRACT

We investigate doubly nonlinear parabolic equations with sign changing solutions. We established the energy estimates to the sign changing solutions within a parabolic domain which is the key elements to determine the regularity results.

Keywords: *Energy Estimate, Cylindrical Domain, p-Laplacian, Parabolic Equation*

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ and for $T > 0$ define the cylindrical doamin $\Omega_T := \Omega \times (0, T]$. Consider the following doubly nonlinear parabolic equation

$$\partial_t(|u|^{p-2}u) - \text{div}\left(\text{D}u|^{p\text{-}2}\text{D}u\right) = 0$$

weakly in Ω_T (1.1)

where $\Delta_p u := \text{div}(|Du|^{p-2}Du)$ is the p-Laplacian. For the case $p = 2$ then this operator transforms to well known heat equation. In this manuscript, the weak solution u is unknown and assumed to be locally bounded, real function which depends on both the time and space variables namely x and t in the cylindrical domain.

In our context, the term structural data indicates the parameters p and dimensional parameter N. It is also assumed that the constant $\gamma > 0$, need to be evaluated quantitatively apriori in terms of the structural data. In addition, denote $\Gamma_T := \partial\Omega_T - \overline{\Omega} \times \{T\}$ to be the parabolic boundary of the

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cylindrical domain Ω_T. For $\theta > 0$, consider the following backward cylinders of the form

$$(x_0, t_0) + Q_\varrho(\theta) = (x_0, t_0) + K_\varrho(0) \times (-\theta\varrho^p, 0] = K_\varrho(x_0) \times (t_0 - \theta\varrho^p, t_0].$$

For the case $\theta = 1$, we will call it as Q_ϱ.

The findings in this article is the following energy inequality.

Theorem 1.1 *Let u be a local weak sub-solution to (1.1).Then there exists a constant $\gamma(p) > 0$ such that for all cylinders $Q_{R,S} = K_R(x_0) \times (t_0 - S, t_0) \Subset \Omega_T$, every $k \in \mathbb{R}$, and every nonnegative, piecewise smooth cut off function ζ vanishing on $\partial K(x_0) \times (t_0 - S, t_0)$, there holds*

$$ess\ sup_{t_0 - S < t < t_0} \int_{K_R(x_0)\times\{t\}} \zeta^p A^{\pm}(k,u)\, dx + \iint_{Q_{R,S}} \zeta^p |D(u-k)_{\pm}|^p\, dx\, dt$$
$$\leqslant \gamma \iint_{Q_{R,S}} [|D\zeta|^p (u-k)_{\pm} + A^{\pm}(k,u)|\partial_t \zeta^p|]\, dx\, dt + \int_{K_R(x_0)\times\{t_0 - S\}} \zeta^p A^{\pm}(k,u)\, dx \qquad (1.2)$$

The definition of a weak solution is given in Definition 2.1.1. Before deriving the estimates we need to be sure that our equation has weak solutions. The existence of global weak solution is established in [25].

1.1 Novelty and Significance.

The equation (1.1) is a standard equation and is known as Trudinger's equation. It is also referred to as a doubly nonlinear parabolic equation due to the non-linearity of both the solution and its spatial gradient. It is of special interest to know why we are taking this type of equation for research as it has a splendid mathematical structure and generate mixed types of degeneracy and/or singularity in partial differential equations, and connection to physical models, including dynamics of glaciers [23], shallow water flows [1, 9, 11] and friction dominated flow in a gas network ([21]). The Trudinger equation is also naturally connected to the non-linear eigenvalue problem $-\Delta_p u = \lambda|u|^{p-2}u$ [22], which plays an essential role in the nonlinear potential theory. The energy estimation of signed solutions studied by V. Bögelein, F. Duzzar and N. Liao in [2] for a more general equations with structure conditions. The energy inequalities of this equation is studied by Trudinger [27], for non-negative weak solutions, analogous to heat equation. The same is analyzed for non-negative weak solutions in [17, 18, 19, 26]. Now its turn to mention our contribution here, we eradicate the restriction of non-negativity of solutions instead we choose sign changing solutions for the energy inequalities to hold. The existence of weak solution to (1.1) is shown in [24]. The energy inequalities for doubly nonlinear equations nonlinear equations has also been studied in [10, 12, 13, 14, 15, 28, 29].

2. Preliminaries

We prepare some notations and technical analysis tools, which are used later [cf. [2], [3], [4], [5], [6], [7], [8], [20], [21]].

2.1 Notation

2.1.1 Notion of Local Weak Solution.

A function

$$u \in C(0,T; L^p_{loc}(\Omega)) \cap L^p_{loc}(0,T; W^{1,p}_{loc}(\Omega))$$

is a local weak sub(super)-solution to (1.1), if for every compact set $C \subset \Omega$ and every sub-interval $[t_1, t_2] \subset (0,T]$

$$\int_C |u|^{p-2}u\zeta dx\Big|_{t_1}^{t_2} + \iint_{C\times(t_1,t_2)} [-|u|^{p-2}u\zeta_t + |Du|^{p-2}Du.D\zeta]dxdt \le (\ge) 0 \qquad (2.1)$$

for all non-negative test functions

$$\zeta \in W^{1,p}_{loc}(0,T; L^p(C)) \cap L^p_{loc}(0,T; W^{1,p}_0(C)).$$

which ensures that all the integrals in (2.1) are convergent. A function u that is both a local weak subsolution and a local weak super-solution to (2.1) is a local weak solution.

2.1.2 Function Spaces on time-space region.

We prepare some function spaces, defined on space-time region. For $1 \le p, q \le \infty$, $L^q(t_1, t_2; L^p(\Omega))$ is a function space of measurable real-valued functions on a space-time region $\Omega \times (t_1, t_2)$ with a finite norm

$$\| v \|_{L^q(t_1,t_2;L^p(\Omega))}: = \begin{cases} \left(\left(\int_{t_1}^{t_2} \| v(t) \|_{L^p(\Omega)}^q dt \right)^{1/q} & (1 \le q < \infty) \\ esssup_{t_1 \le t \le t_2} \| v(t) \|_{L^p(\Omega)} & (q = \infty), \end{cases}$$

where

$$\| v(t) \|_{L^p(\Omega)}: = \begin{cases} \left(\left(\int_{\Omega} |v(x,t)|^p dx \right)^{1/p} & (1 \le p < \infty) \\ esssup_{x \in \Omega} |v(x,t)| & (p = \infty). \end{cases}$$

When $p = q$, we write $L^p(\Omega \times (t_1, t_2)$ $= L^p(t_1, t_2; L^p(\Omega))$ for brevity. For $1 \le p < \infty$ the Sobolev space $W^{1,p}(\Omega)$ is consists of measurable real-valued functions that are weakly differentiable and their weak derivatives are p-th integrable on Ω, with the norm

$$\| v \|_{W^{1,p}(\Omega)}: = \left(\left(\int_{\Omega} |v|^p + |\nabla v|^p dx \right)^{1/p},$$

where $\nabla v = (v_{x_1}, \dots, v_{x_n})$ denotes the gradient of v in a distribution sense, and let $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}}$. Also let $L^q(t_1, t_2; W_0^{1,p}(\Omega))$denote a function space of measurable real-valued functions on space-time region with a finite norm

$$\| v \|_{L^q\left(t_1,t_2;W_0^{1,p}(\Omega)\right)} = \left(\left(\int_{t_1}^{t_2} \| v(t) \|_{W^{1,p}(\Omega)}^q dt \right)^{1/q}.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. For a real number k and for a function v in $L^1(E)$ we define the *truncation* of v by

$$(v - k)_+ \coloneqq \max\{(v - k), 0\}; (k - v)_+: = \max\{(k - v), 0\}. \quad (2.2)$$

For a measurable function v in $L^1(\Omega)$ and a pair of real numbers $k < l$, we set

$$\begin{cases} E \cap \{v > l\}: = \{x \in E: v(x) > l\} \\ E \cap \{v < k\}: = \{x \in E: v(x) < k\} \\ E \cap \{k < v < l\}: = \{x \in E: k < v(x) < l\}. \end{cases} \quad (2.3)$$

2.2 Technical Tools

Following [4], we define the auxiliary function

$$\begin{cases} A^+(k,u): = +(p-1) \int_k^u |s|^{p-2}(s-k)_+ \, ds \\ A^-(k,u): = -(p-1) \int_k^u |s|^{p-2}(s-k)_- \, ds \end{cases} \quad (2.4)$$

for $u, k \in \mathbb{R}$. If $k = 0$, we abbreviate as

$$A^+(u) = A^+(0,u) \quad and \quad A^-(u) = A^-(0,u).$$

It is clear that $A^\pm \geqslant 0$. Now, we will use by embolden $\boldsymbol{b^\alpha}$ to denote the signed α-power of b as

$$\boldsymbol{b^\alpha} = \begin{cases} |b|^{\alpha-1} b, b \neq 0, \\ 0, \quad b = 0. \end{cases}$$

We state a known lemma; cf. [1, Lemma 2.2] for $\alpha \in (0,1)$ and [13, inequality (2.4)] for $\alpha > 1$. It is used to prove the next lemma.

Lemma 2.1 *For any* $\alpha > 0$, *there exists a constant* $\beta = \beta(\alpha)$ *such that, for all* $a, b \in \mathbb{R}$, *the following inequality holds to be valid:*

$$\frac{1}{\beta} |\boldsymbol{b^\alpha} - \boldsymbol{a^\alpha}| \leqslant (|a| + |b|)^{\alpha-1} |b - a| \leqslant \beta |\boldsymbol{b^\alpha} - \boldsymbol{a^\alpha}|.$$

On the basis of the above lemma, we prove the following.

Lemma 2.2 *For all* $w, k \in \mathbb{R}, \alpha > 0$, *there exists a constant* $\beta = \beta(p)$ *such that the following inequality holds to be valid:*

$$\frac{1}{\gamma} (|w| + |k|)^{p-2}(w-k)_\pm^2 \leqslant A^\pm(k,w) \leqslant \gamma(|w| + |k|)^{p-2}(w-k)_\pm^2$$

Proof. We will try to exhibit the proof of A^- and the estimate for the case A^+ is analogous. For $k \leqslant w$, we have $A^-(k,w) = 0 = (w-k)_-$. Therefore, we will consider only $k, w \in \mathbb{R}$ such that $w < k$. Then, we have

$$A^-(k,w) = (p-1) \int_w^k |s|^{p-2}(k-s) ds$$

$$\geqslant (p-1) \int_w^{\frac{1}{2}(k+w)} |s|^{p-2}(k-s) ds$$

$$\geqslant \frac{p-1}{2} (k-w) \int_w^{\frac{1}{2}(k+w)} |s|^{p-2} ds$$

Since $p - 2 > -1$ and as a result the integral on the right hand side exists. Applying the previous lemma, we thus obtain

$$A^-(k,w) \geqslant \frac{1}{2} (k-w) |s|^{p-2} s \Big|_w^{\frac{1}{2}(k+w)}$$

$$\geqslant \frac{1}{\beta(p)} (k-w)(|w| + \frac{1}{2}|k+w|)^{p-2}(\frac{1}{2}(k+w) - w)$$

$$= \frac{1}{2\beta(p)} (k-w)^2(|w| + \frac{1}{2}|k+w|)^{p-2}$$

$$\geqslant \frac{1}{\beta(p)} (k-w)^2(|w| + |k|)^{p-2}$$

In our last computation, we have used the reasoning

$$\frac{1}{2}(|k| + |w|) \leqslant |w| + \frac{1}{2}|k+w| \leqslant 2(|w| + |k|) \quad .$$

This is the lower bound on A^-. Using the same lemma we can obtain

$$A^-(k,w) = (p-1)\int_w^k |s|^{p-2}(k-s)ds$$
$$\leqslant (p-1)(k-w)\int_w^k |s|^{p-2}ds$$
$$= (k-w)|s|^{p-2}s|_w^k$$
$$\leqslant \beta(p)(k-w)^2(|w|+|k|)^{p-2}$$

This completes the proof.

For the time regularity of the solution u, we define the following mollification in time:

$[u]_h(x,t) \overset{def}{=} \frac{1}{h}\int_0^t e^{\frac{s-t}{h}} u(x,s)ds$ for any$u \in L^1(\Omega_T)$ (2.5)

Lemma 2.3 (Properties of mollification) *[16]*

1. If $u \in L^p(\Omega_T)$ then $\| [u]_h(x,t) \|_{L^p(\Omega_T)} \leqslant \| u \|_{L^p(\Omega_T)}$ and
$$\frac{\partial [u]_h}{\partial t} = \frac{u-[u]_h}{h} \in L^p(\Omega_T).$$
Moreover, $[u]_h \to u$ in $L^p(\Omega_T)$ as $h \to 0$.
2. If, in addition, $\nabla([u]_h) = [\nabla u]_h$ componentwise,
$\| \nabla([u]_h) \|_{L^p(\Omega_T)} \leqslant \| \nabla u \|_{L^p(\Omega_T)}$
and $\nabla[u]_h \to \nabla u$ in $L^p(\Omega_T)$ as $h \to 0$.
3. Furthermore, if $u_k \to u$ in $L^p(\Omega_T)$ then also
$$[u_k]_h \to [u]_h \ and\ \frac{\partial [u_k]_h}{\partial t} \to \frac{\partial [u]_h}{\partial t}$$
in $L^p(\Omega_T)$. and $\nabla[u]_h \to \nabla u$ in $L^p(\Omega_T)$ as $h \to 0$.
4. If $\nabla u_k \to \nabla u$ in $L^p(\Omega_T)$ then also $\nabla[u_k]_h \to \nabla[u]_h \in L^p(\Omega_T)$.
5. Analogous results hold for weak convergence in $L^p(\Omega_T)$.
6. Finally, if$\varphi \in C(\overline{\Omega}_T)$ then $[\varphi]_h(x,t) + e^{-\frac{t}{h}}\varphi(x,0) \to \varphi(x,t)$ uniformly in Ω_T as $h \to 0$.

3 Energy Estimates

In this section we derive the main theorem. Here we consider the sign changing solutions. Sub-solutions and super-solutions plays different role which will be emphasize here. When we write "u is a sub(super)-solution..." and use "$\pm$"or "$\mp$" in what follows, we mean the sub-solution corresponds to the upper sign and the super-solution corresponds to the lower sign in the statement. First of all, we state the energy estimates and present the proof for sub-solution case and the super-solution case is similar and is not presented here.

Proposition 3.1 *Let u be a local weak sub-solution to (1.1).Then there exists a constant $\gamma(p) > 0$ such that for all cylinders $Q_{R,S} = K_R(x_0) \times (t_0 - S, t_0) \Subset \Omega_T$, every $k \in \mathbb{R}$, and every nonnegative, piecewise smooth cut off function ζ vanishing on $\partial K(x_0) \times (t_0 - S, t_0)$, there holds*

$$ess\ sup_{t_0-S<t<t_0} \int_{K_R(x_0)\times\{t\}} \zeta^p A^\pm(k,u)\,dx + \iint_{Q_{R,S}} \zeta^p |D(u-k)_\pm|^p\,dx\,dt$$
$$\leqslant \gamma \iint_{Q_{R,S}} [|D\zeta|^p(u-k)_\pm + A^\pm(k,u)|\partial_t \zeta^p|]\,dx\,dt$$
$$+ \int_{K_R(x_0)\times\{t_0-S\}} \zeta^p A^\pm(k,u)\,dx \qquad (3.1)$$

Proof. By definition of weak solution u to (1.1), we have

$$\iint_{\Omega_T} \partial_t(|u|^{p-2}u)\varphi\,dx\,d + \iint_{\Omega_T} |Du|^{p-2}Du.D\varphi\,dx\,dt \leqslant 0$$

Let us choos $\varphi(x,t) = \frac{1}{h}\int_t^T e^{\frac{t-s}{h}}\psi(x,s)\,ds$ and using it to the above inequality gives

$$\iint_{\Omega_T} \left[\partial_t(|u|^{p-2}u)\frac{1}{h}\int_t^T e^{\frac{t-s}{h}}\psi(x,s)\,ds\right]dx\,dt + \iint_{\Omega_T} \left[|Du|^{p-2}Du.\frac{1}{h}\int_t^T e^{\frac{t-s}{h}}D\psi(x,s)\,ds\right]dx\,dt \leqslant 0$$

At first handle the 1st part of the above inequality, we have

$$\iint_{\Omega_T} \left[\partial_t(|u|^{p-2}u)\frac{1}{h}\int_t^T e^{\frac{t-s}{h}}\psi(x,s)\,ds\right]dx\,dt$$
$$= \int_\Omega \int_0^T \partial_t[u^{p-1}]_h(x,s)\psi(x,s)\,ds\,dx$$
$$- \int_\Omega u^{p-1}(x,0)\int_0^T \frac{1}{h}e^{\frac{-s}{h}}\psi(x,s)\,ds\,dx$$

Now manipulating the second term gives us

$$\iint_{\Omega_T} \left[|Du|^{p-2}Du\,\frac{1}{h}\int_t^T e^{\frac{t-s}{h}}D\psi(x,s)\,ds\right]dx\,dt$$
$$= \int_\Omega \int_0^T \left[\frac{1}{h}\int_0^s \left(|Du|^{p-2}Du\,e^{\frac{t-s}{h}}\right)dt\right]D\psi\,ds\,dx$$
$$= \int_\Omega \int_0^T [(Du)^{p-1}]_h D\psi(x,s)\,ds\,dx$$

Aggregating the above two terms then

$$\int_\Omega \int_0^T \partial_t [u^{p-1}]_h(x,s)\psi(x,s)\, ds\, dx + \int_\Omega \int_0^T [(Du)^{p-1}]_h D\psi(x,s)\, ds\, dx$$

$$\leqslant \int_\Omega u^{p-1}(x,0) \int_0^T \frac{1}{h} e^{\frac{-s}{h}} \psi(x,s)\, ds\, dx \qquad (3.2)$$

Now we will choose the following setting $Q_{R,S} = K_R(x_0) \times (t_0 - S, t_0] \Subset \Omega_T$, $\zeta \in C^1(Q_{R,S}, [0,1])$be a cut off function such that $\zeta = 0$ on $\partial K_R(x_0) \times (t_0 - S, t_0)$ and $\psi_\epsilon \in W^{1,\infty}((t_0 - S, t_0), [0,1])$such that

$$\psi_\epsilon(t) := \begin{cases} 0: & t_0 - S \leqslant t \leqslant t_1 - \varepsilon \\ 1 + \dfrac{t - t_1}{\varepsilon}: & t_1 - \varepsilon < t \leqslant t_1 \\ 1: & t_1 < t \leqslant t_2 \\ 1 - \dfrac{t - t_2}{\varepsilon}: & t_2 < t \leqslant t_2 + \varepsilon \\ 0: & t_2 + \varepsilon < t \leqslant t_0 \end{cases}$$

To get the desired estimates, let $\psi(x,t) = \zeta^p(x,t)\psi_\epsilon(t)(u(x,t) - k)_+$

After using (3.2), we have

$$\int_{\Omega_S} \partial_t [u^{p-1}]_h(x,s)\zeta^p(x,s)\psi_\varepsilon(s)(u(x,s) - k)_+\, ds\, dx$$

$$+ \int_{\Omega_S} [(Du)^{p-1}]_h D(\zeta^p(x,s)\psi_\varepsilon(s)(u(x,s) - k)_+))\, ds\, dx$$

$$\leqslant \int_\Omega u^{p-1}(x,0) \frac{1}{h} \int_0^T e^{\frac{-s}{h}} \zeta^p(x,s)\psi_\varepsilon(s)(u(x,s) - k)_+\, ds\, dx \qquad (3.3)$$

Treating each of the terms above separately gives

$$\int_{\Omega_S} \partial_t [u^{p-1}]_h(x,s)\zeta^p(x,s)\psi_\varepsilon(s)(u(x,s) - k)_+\, ds\, dx$$

$$\leq -\iint_{Q_{R,S}} (\zeta^p \psi_\varepsilon' + \psi_\varepsilon \partial_t \zeta^p) A^+\left(k, [u^{p-1}]_h^{\frac{1}{p-1}}\right) dx\, ds \qquad (3.4)$$

Now here aim to show that $[u^{p-1}]_h \to u^{p-1}$ in $L^{\frac{p}{p-1}}(\Omega_T)$ as $h \to 0$

$$\iint_{\Omega_T} ([u^{p-1}]_h - u^{p-1})^{\frac{p}{p-1}}\, dx\, dt$$

$$\leqslant \iint_{\Omega_T} \left\{ \int_{-\frac{t}{h}}^0 e^\tau (u^{p-1}(t + \tau h) - u^{p-1}(t))\, d\tau + \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} u^{p-1}(x,t) ds \right\}^{\frac{p}{p-1}} dx\, dt$$

$$\leqslant \iint_{\Omega_T} \left(\int_{-\frac{t}{h}}^0 e^\tau (u^{p-1}(t + \tau h) - u^{p-1}(t))\, d\tau \right)^{\frac{p}{p-1}} dx\, dt + \iint_{\Omega_T} \left(\frac{1}{h} \int_0^t e^{\frac{s-t}{h}} u^{p-1}(x,t) ds \right)^{\frac{p}{p-1}} dx\, dt$$

$$= \iint_{\Omega_T} \left(\int_{-\frac{t}{h}}^0 e^\tau (u^{p-1}(t + \tau h) - u^{p-1}(t))\, d\tau \right)^{\frac{p}{p-1}} dx\, dt + \iint_{\Omega_T} \left(u^{p-1} - u^{p-1} e^{-\frac{t}{h}} \right)^{\frac{p}{p-1}} dx\, dt \qquad (3.5)$$

Thus imposing the limit $h \downarrow 0$ provide us the claim. Therefore, applying the claim just proved above

$$\liminf_{h\downarrow 0} \iint_{Q_{R,S}} \partial_t [u^{p-1}]_h \psi(x,s)\, dx\, ds \geqslant -\iint_{Q_{R,S}} (\zeta^p \psi_\varepsilon' - \psi_\varepsilon \partial_t \zeta^p) A^+(k,u)\, dx\, ds =: -[I_1 + I_2]$$

We now pass to the limit $\varepsilon \downarrow 0$. For the term I_1, we have for any$t_0 - S < t_1 < t_2 < t_0$

$$\lim_{\varepsilon\downarrow 0} I_1 = \int_{K_R} \psi^p(x,t_1) A^+(k, u(x,t_1))\, dx - \int_{K_R} \psi^p(x,t_2) A^+(k, u(x,t_2))\, dx$$

For I_2 we get

$$\lim_{\varepsilon\downarrow 0} I_2 = \iint_{K_R \times (t_1,t_2)} \partial_t \zeta^p A^+(k, u(x,t))\, dx\, ds$$

Thus we have,

$$\iint_{Q_{R,S}} \partial_t [u^{p-1}]_h \psi(x,s)\, dx\, ds \geqslant \int_{K_R} \zeta^p(x,t_2) A^+\big(k, u(x,t_2)\big) d - \int_{K_R} \zeta^p(x,t_1) A^+(k, u(x,t_1)\, dx$$

$$+ \iint_{K_R \times (t_1,t_2)} \partial_t \zeta^p A^+(k, u(x,t))\, dx\, ds \qquad (3.6)$$

Now we show that the boundary term disappears as $h \downarrow 0 \lim_{h\downarrow 0} \int_\Omega u^{p-1}(x,0) \left(\int_0^T \frac{1}{h} e^{\frac{-s}{h}} \psi(x,s)\, ds \right) dx = 0$

Now

$$\lim_{h\downarrow 0}\frac{1}{h}\int_0^T e^{\frac{-s}{h}}\psi(x,s)\,ds-\psi(x,0)$$

$$=\lim_{h\downarrow 0}\left\{\frac{1}{h}\int_0^T e^{\frac{-s}{h}}(\psi(x,s)-\psi(x,0))\,ds+\frac{1}{h}\int_0^T e^{\frac{-s}{h}}\psi(x,0)\,ds\right\}$$

$$=\lim_{h\downarrow 0}\left\{\frac{1}{h}\int_0^{T/h} e^{-\tau}(\psi(x,s)-\psi(x,0))h\,d\tau\right\}+\lim_{h\downarrow 0}\frac{1}{h}e^{\frac{-s}{h}}\Big|_0^T\psi(x,0)$$

$$=\left\{\int_0^{T/h} e^{-\tau}(\psi(x,s)-\psi(x,0))h\,d\tau\right\}+\lim_{h\downarrow 0}(1-e^{\frac{-T}{h}})\psi(x,0)$$

It's turn to treat the diffusion term. After imposing the limit $h\downarrow 0$, we utilize the ellipticity and growth assumptions and thereafter apply Young's inequality to the integral containing the terms $(u-k)_+$ and $D(u-k)_+$. Hence we can compute the following

$$\lim_{h\downarrow 0}\iint_{\Omega_T}[(Du)^{p-1}]_h\psi(x,t)\,dt\,dx=$$

$$\lim_{h\downarrow 0}\iint_{\Omega_T}\left(\frac{1}{h}\int_0^t e^{\frac{s-t}{h}}|Du|^{p-2}Du\,ds\right)\psi(x,t)\,dt\,dx$$

$$=\iint_{\Omega_T}\left(\lim_{h\downarrow 0}\frac{1}{h}\int_{-\frac{t}{h}}^0 e^{\tau}Du(x,t+u\tau)^{p-1}\,d\tau\right)dx\,dt$$

$$=\iint_{\Omega_T}\lim_{h\downarrow 0}\left\{\int_{-\frac{t}{h}}^0 e^{\tau}[Du(x,t+u\tau)^{p-1}]\,d+Du(x,t)^{p-1}\int_{-\frac{t}{h}}^0 e^{\tau}\,d\tau\right\}dx\,dt$$

As it is known that if $f\in L^1$ then $\int_0^T[f(t+h)-f(t)]\,dz\searrow 0$ as $h\downarrow 0$ as well as

$$\lim_{h\downarrow 0}\iint_{\Omega_T}[Du]_h^{p-1}\psi(x,t)\,dt\,dx=\iint_{\Omega_T}Du(x,t)^{p-1}\,dt\,dx$$

Thus the diffusion term has the following computation $\iint_{\Omega_S}Du(x,s)^{p-1}D\psi(x,s)\,dx\,ds$

$$=\iint_{\Omega_S}Du(x,s)^{p-1}D(\zeta^p\psi_\varepsilon(u-k)+)\,dx\,ds$$

$$\geqslant\iint_{\Omega_S}D(u(x,s)-k)_+^p\zeta^p\psi_\varepsilon\,dx\,ds-\gamma\iint_{\Omega_S}Du(x,s)^{p-1}|D\zeta)^p(u-k)_+^p\psi_\varepsilon\,dx\,ds \quad (3.7)$$

Combining the above estimates as $\varepsilon\downarrow 0$, we have

$$\int_{K_R\times\{t_2\}}\zeta^pA^+(k,u)\,dx+\iint_{K_R\times(t_1,t_2)}\zeta^p|D(u-k)_+|^p\,dx\,ds$$

$$\leqslant\iint_{K_R\times(t_1,t_2)}[\gamma|D\zeta|^p(u-k)_+^pA^+(k,u)|\partial_t\zeta^p|]\,dx\,dt+\int_{K_R\times\{t_1\}}\zeta^pA^+(k,u(x,t_1))\,dx \quad (3.8)$$

whenever $t_0-S<t_1<t_2<t_0$. The constant $\gamma=\gamma(p)$. Since $u\in C(0,T;L^p_{loc}(\Omega))$. Thus we have $ess\ sup_{t_0-S<t<t_0}\int_{K_R(x_0)\times\{t\}}\zeta^pA^+(k,u)\,dx+\iint_{K_R\times(t_0-S,t_0)}\zeta^p|D(u-k)_+|^p\,dx\,dt$

$$\leqslant\iint_{K_{R,S}}[\gamma|D\zeta|^p(u-k)_+^p+A^+(k,u)|\partial_t\zeta^p|]\,dx\,dt+\int_{K_R(x_0)\times\{t_0-S\}}\zeta^pA^+(k,u)\,dx \quad (3.9)$$

In the above approximation we have taken the essential supremum with respect to $t_2\in(t_0-S,t_0)$. After discarding the second integral from left side which leads to an estimate of the essential supremum of the first integral. Similarly, discarding the first integral and passing to the limit $t_2\uparrow t_0$ we arrive at the estimate for the second integral on the left side.

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